Two-parameter Quantum Group of Exceptional Type G_2 and Lusztig's Symmetries

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Abstract. We give the defining structure of two-parameter quantum group of type G_2 defined over a field $\mathbb{Q}(r,s)$ (with $r \neq s$), and prove the Drinfel'd double structure as its upper and lower triangular parts, extending an earlier result of [BW1] in type A and a recent result of [BGH1] in types B, C, D. We further discuss the Lusztig's $\mathbb{Q}$-isomorphisms from $U_{r,s}(G_2)$ to its associated object $U_{s^{-1},r^{-1}}(G_2)$, which give rise to the usual Lusztig's symmetries defined not only on $U_q(G_2)$ but also on the centralized quantum group $U_q^c(G_2)$ only when $r = s^{-1} = q$. (This also reflects the distinguishing difference between our newly defined two-parameter object and the standard Drinfel'd-Jimbo quantum groups). Some interesting (r,s)-identities holding in $U_{r,s}(G_2)$ are derived from this discussion.

1. Two-parameter Quantum Group $U_{r,s}(G_2)$

Let $\mathbb{K} = \mathbb{Q}(r,s)$ be a field of rational functions with two indeterminates r, s.

Let Φ be a finite root system of G_2 with Π a base of simple roots, which is a subset of a Euclidean space $E = \mathbb{R}^3$ with an inner product $(\, ,\,)$. Let ϵ_1, ϵ_2, ϵ_3 denote an orthonormal basis of E, then $\Pi = \{\alpha_1 = \epsilon_1 - \epsilon_2,\ \alpha_2 = \epsilon_2 + \epsilon_3 - 2\epsilon_1\}$ and $\Phi = \pm\{\alpha_1, \alpha_2, \alpha_2 + \alpha_1, \alpha_2 + 2\alpha_1, \alpha_2 + 3\alpha_1, 2\alpha_2 + 3\alpha_1\}$. In this case, we set $r_1 = r^{\frac{(\alpha_1,\,\alpha_1)}{2}} = r,\ r_2 = r^{\frac{(\alpha_2,\,\alpha_2)}{2}} = r^3$ and $s_1 = s^{\frac{(\alpha_1,\,\alpha_1)}{2}} = s,\ s_2 = s^{\frac{(\alpha_2,\,\alpha_2)}{2}} = s^3$.

We begin by giving the definition of two-parameter quantum group of type G_2, which is new.

Definition 1.1. Let $U = U_{r,\,s}(G_2)$ be the associative algebra over $\mathbb{Q}(r,s)$ generated by symbols $e_i,\ f_i,\ \omega_i^{\pm1},\ \omega_i'^{\pm1}$ $(1 \le i \le 2)$ subject to the relations

$(G1)$ $[\,\omega_i^{\pm1}, \omega_j^{\pm1}\,] = [\,\omega_i^{\pm1}, \omega_j'^{\pm1}\,] = [\,\omega_i'^{\pm1}, \omega_j'^{\pm1}\,] = 0, \quad \omega_i\omega_i^{-1} = 1 = \omega_j'\omega_j'^{-1}.$

$(G2)$
$$\begin{aligned}
\omega_1\, e_1\, \omega_1^{-1} &= (rs^{-1})\, e_1, & \omega_1\, f_1\, \omega_1^{-1} &= (r^{-1}s)\, f_1,\\
\omega_1\, e_2\, \omega_1^{-1} &= s^3\, e_2, & \omega_1\, f_2\, \omega_1^{-1} &= s^{-3}\, f_2,\\
\omega_2\, e_1\, \omega_2^{-1} &= r^{-3}\, e_1, & \omega_2\, f_1\, \omega_2^{-1} &= r^3\, f_1,\\
\omega_2\, e_2\, \omega_2^{-1} &= (r^3s^{-3})\, e_2, & \omega_2\, f_2\, \omega_2^{-1} &= (r^{-3}s^3)\, f_2.
\end{aligned}$$

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$$(G3)\qquad \begin{aligned} \omega_1'\, e_1\, \omega_1'^{-1} &= (r^{-1}s)\, e_1, & \omega_1'\, f_1\, \omega_1'^{-1} &= (rs^{-1})\, f_1,\\ \omega_1'\, e_2\, \omega_1'^{-1} &= r^3\, e_2, & \omega_1'\, f_2\, \omega_1'^{-1} &= r^{-3}\, f_2,\\ \omega_2'\, e_1\, \omega_2'^{-1} &= s^{-3}\, e_1, & \omega_2'\, f_1\, \omega_2'^{-1} &= s^3\, f_1,\\ \omega_2'\, e_2\, \omega_2'^{-1} &= (r^{-3}s^3)\, e_2, & \omega_2'\, f_2\, \omega_2'^{-1} &= (r^3s^{-3})\, f_2. \end{aligned}$$

$(G4)$ For $1 \le i,\, j \le 2$, we have

$$[\, e_i, f_j\,] = \delta_{ij}\frac{\omega_i - \omega_i'}{r_i - s_i}.$$

$(G5)$ ($(r,\, s)$-Serre relations)

$$(G5)_1\qquad e_2^2e_1 - (r^{-3}+s^{-3})\, e_2e_1e_2 + (rs)^{-3}\, e_1e_2^2 = 0,$$

$$(G5)_2\qquad \begin{aligned} &e_1^4e_2 - (r{+}s)(r^2+s^2)\, e_1^3e_2e_1 + rs(r^2+s^2)(r^2+rs+s^2)\, e_1^2e_2e_1^2\\ &\quad -(rs)^3(r+s)(r^2+s^2)\, e_1e_2e_1^3 + (rs)^6e_2e_1^4 = 0. \end{aligned}$$

$(G6)$ ($(r,\, s)$-Serre relations)

$$(G6)_1\qquad f_1f_2^2 - (r^{-3}+s^{-3})\, f_2f_1f_2 + (rs)^{-3}\, f_2^2f_1 = 0,$$

$$(G6)_2\qquad \begin{aligned} &f_2f_1^4 - (r{+}s)(r^2+s^2)\, f_1f_2f_1^3 + rs(r^2+s^2)(r^2+rs+s^2)\, f_1^2f_2f_1^2\\ &\quad -(rs)^3(r+s)(r^2+s^2)\, f_1^3f_2f_1 + (rs)^6\, f_1^4f_2 = 0. \end{aligned}$$

PROPOSITION 1.2. *The algebra $U_{r,s}(G_2)$ is a Hopf algebra with comultiplication, counit and antipode given by*

$$\begin{aligned} \Delta(\omega_i^{\pm1}) &= \omega_i^{\pm1}\otimes\omega_i^{\pm1}, & \Delta(\omega_i'^{\pm1}) &= \omega_i'^{\pm1}\otimes\omega_i'^{\pm1},\\ \Delta(e_i) &= e_i\otimes 1 + \omega_i\otimes e_i, & \Delta(f_i) &= 1\otimes f_i + f_i\otimes\omega_i',\\ \varepsilon(\omega_i^{\pm}) &= \varepsilon(\omega_i'^{\pm1}) = 1, & \varepsilon(e_i) &= \varepsilon(f_i) = 0,\\ S(\omega_i^{\pm1}) &= \omega_i^{\mp1}, & S(\omega_i'^{\pm1}) &= \omega_i'^{\mp1},\\ S(e_i) &= -\omega_i^{-1}e_i, & S(f_i) &= -f_i\,\omega_i'^{-1}. \end{aligned}$$

□

REMARK 1.3. (I) When $r = q = s^{-1}$, the quotient Hopf algebra of $U_{r,\,s}(G_2)$ modulo the Hopf ideal generated by elements $\omega_i' - \omega_i^{-1}(1 \le i \le 2)$, is just the standard quantum group $U_q(G_2)$ of Drinfel'd-Jimbo type; the quotient modulo the Hopf ideal generated by elements $\omega_i' - z_i\omega_i^{-1}$ $(1 \le i \le 2)$, where z_i runs over the center, is the *centralized quantum group* $U_q^c(G_2)$.

(II) In any Hopf algebra $\mathcal{H}$, there exist the left-adjoint and the right-adjoint action defined by the Hopf algebra structure:

$$\mathrm{ad}_l\, a\,(b) = \sum_{(a)} a_{(1)}\, b\, S(a_{(2)}), \qquad \mathrm{ad}_r\, a\,(b) = \sum_{(a)} S(a_{(1)})\, b\, a_{(2)},$$

where $\Delta(a) = \sum_{(a)} a_{(1)}\otimes a_{(2)} \in \mathcal{H}\otimes\mathcal{H}$, for any $a,\, b \in \mathcal{H}$.

From the viewpoint of adjoint actions, the (r,s)-Serre relations $(G5)$, $(G6)$ take on the simpler forms

$$\begin{aligned} \big(\mathrm{ad}_l\, e_i\big)^{1-a_{ij}}\,(e_j) &= 0, \qquad \textit{for any}\ \ i \ne j,\\ \big(\mathrm{ad}_r\, f_i\big)^{1-a_{ij}}\,(f_j) &= 0, \qquad \textit{for any}\ \ i \ne j. \end{aligned}$$

2. Drinfel'd Quantum Double

Definition 2.1. A (Hopf) dual pairing of two Hopf algebras $\mathcal{A}$ and $\mathcal{U}$ (see [BGH1], or [KS]) is a bilinear form $\langle\,,\rangle : \mathcal{U}\times\mathcal{A}\longrightarrow\mathbb{K}$ such that

$$\langle f,\,1_{\mathcal{A}}\rangle=\varepsilon_{\mathcal{U}}(f),\qquad \langle 1_{\mathcal{U}},\,a\rangle=\varepsilon_{\mathcal{A}}(a),\tag{1}$$

$$\langle f,\,a_1a_2\rangle=\langle\triangle_{\mathcal{U}}(f),\,a_1\otimes a_2\rangle,\qquad \langle f_1f_2,\,a\rangle=\langle f_1\otimes f_2,\,\triangle_{\mathcal{A}}(a)\rangle,\tag{2}$$

for all $f,\,f_1,\,f_2\in\mathcal{U}$, and $a,\,a_1,\,a_2\in\mathcal{A}$, where $\varepsilon_{\mathcal{U}}$ and $\varepsilon_{\mathcal{A}}$ denote the counits of $\mathcal{U}$ and $\mathcal{A}$, respectively, and $\triangle_{\mathcal{U}}$ and $\triangle_{\mathcal{A}}$ are their comultiplications.

A direct consequence of the defining properties above is that

$$\langle S_{\mathcal{U}}(f),\,a\rangle=\langle f,\,S_{\mathcal{A}}(a)\rangle,\quad f\in\mathcal{U},\,a\in\mathcal{A},$$

where $S_{\mathcal{U}},S_{\mathcal{A}}$ denote the respective antipodes of $\mathcal{U}$ and $\mathcal{A}$.

Definition 2.2. A bilinear form $\langle\,,\rangle : \mathcal{U}\times\mathcal{A}\longrightarrow\mathbb{K}$ is called a skew-dual pairing of two Hopf algebras $\mathcal{A}$ and $\mathcal{U}$ (see [BGH1]) if $\langle\,,\rangle : \mathcal{U}^{\rm cop}\times\mathcal{A}\longrightarrow\mathbb{K}$ is a Hopf dual pairing of $\mathcal{A}$ and $\mathcal{U}^{\rm cop}$, where $\mathcal{U}^{\rm cop}$ is the Hopf algebra having the opposite comultiplication to $\mathcal{U}$, and $S_{\mathcal{U}^{\rm cop}}=S_{\mathcal{U}}^{-1}$ is invertible.

Denote by $\mathcal{B}=B(G_2)$ the Hopf subalgebra of $U_{r,\,s}(G_2)$ generated by $e_j,\omega_j^{\pm1}$ and by $\mathcal{B}'=B'(G_2)$ the one generated by $f_j,\omega_j'^{\pm1}$, where $1\le j\le 2$.

Proposition 2.3. *There exists a unique skew-dual pairing $\langle\,,\rangle : \mathcal{B}'\times\mathcal{B}\longrightarrow\mathbb{Q}(r,\,s)$ of the Hopf subalgebras $\mathcal{B}$ and $\mathcal{B}'$, such that*

$$\langle f_i,\,e_j\rangle=\delta_{ij}\frac{1}{s_i-r_i},\qquad (1\le i,\,j\le 2),\tag{3}$$

$$\langle\omega_1',\,\omega_1\rangle=rs^{-1},\tag{4_1}$$

$$\langle\omega_1',\,\omega_2\rangle=r^{-3},\tag{4_2}$$

$$\langle\omega_2',\,\omega_1\rangle=s^{3},\tag{4_3}$$

$$\langle\omega_2',\,\omega_2\rangle=r^{3}s^{-3},\tag{4_4}$$

$$\langle\omega_i'^{\pm1},\,\omega_j^{-1}\rangle=\langle\omega_i'^{\pm1},\,\omega_j\rangle^{-1}=\langle\omega_i',\,\omega_j\rangle^{\mp1},\qquad(1\le i,\,j\le 2),\tag{5}$$

and all other pairs of generators are 0. *Moreover, we have $\langle S(a),\,S(b)\rangle=\langle a,\,b\rangle$ for $a\in\mathcal{B}',b\in\mathcal{B}$.*

Proof. Since any skew-dual pairing of bialgebras is determined by its values on generators, uniqueness is clear. We proceed to prove the existence of the pairing.

We begin by defining a bilinear form $\langle\,,\rangle : \mathcal{B}'^{\rm cop}\times\mathcal{B}\to\mathbb{Q}(r,\,s)$ first on the generators satisfying (3), (4), and (5). Then we extend it to a bilinear form on $\mathcal{B}'^{\rm cop}\times\mathcal{B}$ by requiring that (1) and (2) hold for $\triangle_{\mathcal{B}'^{\rm cop}}=\triangle_{\mathcal{B}'}^{\rm op}$. We will verify that the relations in $\mathcal{B}$ and $\mathcal{B}'$ are preserved, ensuring that the form is well-defined and so is a dual pairing of $\mathcal{B}$ and $\mathcal{B}'^{\rm cop}$ by definition.

It is direct to check that the bilinear form preserves all the relations among the $\omega_i^{\pm1}$ in $\mathcal{B}$ and the $\omega_i'^{\pm1}$ in $\mathcal{B}'$. Next, the structure constants (4_n) ensure the compatibility of the form defined above with those relations of $(G2)$ and $(G3)$ in $\mathcal{B}$ or $\mathcal{B}'$ respectively. We are left to verify that the form preserves the $(r,\,s)$-Serre relations in $\mathcal{B}$ and $\mathcal{B}'$. It suffices to show that the form on $\mathcal{B}'^{\rm cop}\times\mathcal{B}$ preserves the (r,s)-Serre relations in $\mathcal{B}$; the verification for $\mathcal{B}'^{\rm cop}$ is similar.

First, let us show that the form preserves the $(r,\,s)$-Serre relation of degree 2 in $\mathcal{B}$, that is,

$$\langle X,\, e_2^2e_1-(r^{-3}+s^{-3})\,e_2e_1e_2+r^{-3}s^{-3}\,e_1e_2^2\,\rangle=0,$$

where X is any word in the generators of $\mathcal{B}'$. It suffices to consider three monomials: $X=f_2^2f_1,\ f_2f_1f_2,\ f_1f_2^2$. However, in the degree 2's situation for type G_2, its proof is the same as that of type C_2 (see [BGH1, (7C) and thereafter]).

Next, we verify that the $(r,\,s)$-Serre relation of degree 4 in $\mathcal{B}$ is preserved by the form, that is, we show that

$$\begin{aligned}\langle X,\ e_1^4e_2-(r+s)(r^2+s^2)\,e_1^3e_2e_1+rs(r^2+s^2)(r^2+rs+s^2)\,e_1^2e_2e_1^2\\ -(rs)^3(r+s)(r^2+s^2)\,e_1e_2e_1^3+\ (rs)^6\,e_2e_1^4\,\rangle,\end{aligned}$$

vanishes, where X is any word in the generators of $\mathcal{B}'$. By definition, this expression equals

$$(*)\qquad \begin{aligned}\langle\triangle^{(4)}(X),\ & e_1\otimes e_1\otimes e_1\otimes e_1\otimes e_2\\ & -(r+s)(r^2+s^2)\,e_1\otimes e_1\otimes e_1\otimes e_2\otimes e_1\\ & +rs(r^2+s^2)(r^2+rs+s^2)\,e_1\otimes e_1\otimes e_2\otimes e_1\otimes e_1\\ & -(rs)^3(r+s)(r^2+s^2)\,e_1\otimes e_2\otimes e_1\otimes e_1\otimes e_1\\ & +(rs)^6\,e_2\otimes e_1\otimes e_1\otimes e_1\otimes e_1\,\rangle,\end{aligned}$$

where $\triangle$ in the left-hand side of the pairing $\langle\,,\rangle$ indicates $\triangle_{\mathcal{B}'}^{\mathrm{op}}$. In order for any one of these terms to be nonzero, X must involve exactly four f_1 factors, one f_2 factor, and arbitrarily many $\omega_j'^{\pm1}$ factors $(j=1,2)$.

It suffices to consider five key cases:

(i) $X=f_1^4f_2$, we have

$$\begin{aligned}\triangle^{(4)}(X)=\big(&\omega_1'\otimes\omega_1'\otimes\omega_1'\otimes\omega_1'\otimes f_1+\omega_1'\otimes\omega_1'\otimes\omega_1'\otimes f_1\otimes 1\\ &+\omega_1'\otimes\omega_1'\otimes f_1\otimes 1\otimes 1+\omega_1'\otimes f_1\otimes 1\otimes 1\otimes 1+f_1\otimes 1\otimes 1\otimes 1\otimes 1\big)^4\cdot\\ &\big(\omega_2'\otimes\omega_2'\otimes\omega_2'\otimes\omega_2'\otimes f_2+\omega_2'\otimes\omega_2'\otimes\omega_2'\otimes f_2\otimes 1\\ &+\omega_2'\otimes\omega_2'\otimes f_2\otimes 1\otimes 1+\omega_2'\otimes f_2\otimes 1\otimes 1\otimes 1+f_2\otimes 1\otimes 1\otimes 1\otimes 1\big).\end{aligned}$$

Expanding $\triangle^{(4)}(X)$, we get 120 relevant terms having a nonzero contribution to $(*)$. They are listed in TABULAR 1 of Appendix, together with their pairing values, where we have introduced

$$a=\langle f_1,\,e_1\rangle^4\langle f_2,\,e_2\rangle,\quad x=\langle\omega_1',\,\omega_1\rangle,\quad \bar{x}=\langle\omega_1',\,\omega_2\rangle,\quad y=\langle\omega_2',\,\omega_1\rangle.$$

The expression in $(*)$ equals

$$\begin{aligned}&(\text{sum of expressions in column 1})\\ &-(\text{sum of expressions in column 2})\cdot(r+s)(r^2+s^2)\\ &+(\text{sum of expressions in column 3})\cdot rs(r^2+s^2)(r^2+rs+s^2)\\ &-(\text{sum of expressions in column 4})\cdot(rs)^3(r+s)(r^2+s^2)\\ &+(\text{sum of expressions in column 5})\cdot(rs)^6.\end{aligned}$$

Thus, if we sum up all the pairing values listed in each column of TABULAR 1 and multiply by the appropriate factor, we obtain the paring value of $(*)$:

$$
\begin{aligned}
& a(1+3x+5x^2+6x^3+5x^4+3x^5+x^6)\cdot[\,1-(r+s)(r^2+s^2)\bar{x} \\
&\quad +rs(r^2+s^2)\cdot(r^2+rs+s^2)\bar{x}^2-(rs)^3(r+s)(r^2+s^2)\bar{x}^3+(rs)^6\bar{x}^4\,] \\
&= a(1+3x+5x^2+6x^3+5x^4+3x^5+x^6)(1-r^3\bar{x})(1-r^2s\bar{x})(1-rs^2\bar{x})(1-s^3\bar{x}) \\
&= 0 \qquad\qquad (\text{because } \bar{x}=\langle\omega_1',\,\omega_2\rangle=r^{-3}\,).
\end{aligned}
$$

(ii) $X=f_2f_1^4$. By calculation, we get 120 relevant terms of $\triangle^{(4)}(X)$ in $(*)$ and their pairing values listed in TABULAR 2 of Appendix.

If we sum up all the pairing values listed in each column of TABULAR 2, then we obtain the paring values of $(*)$:

$$
\begin{aligned}
& a(1+3x+5x^2+6x^3+5x^4+3x^5+x^6)\cdot[\,y^4-(s^3+rs^2+r^2s+r^3)\cdot y^3 \\
&\quad +rs(s^4+rs^3+2r^2s^2+r^3s+r^4)\cdot y^2-(rs)^3(s^3+rs^2+r^2s+r^3)\cdot y+(rs)^6\,] \\
&= a(1+3x+5x^2+6x^3+5x^4+3x^5+x^6)(y-r^3)(y-r^2s)(y-rs^2)(y-s^3) \\
&= 0 \qquad\qquad (\text{because } y=\langle\omega_2',\,\omega_1\rangle=s^3).
\end{aligned}
$$

(iii) $X=f_1^2f_2f_1^2$. By calculation, we get 120 relevant of $\triangle^{(4)}(X)$ in $(*)$ and their pairing values listed in TABULAR 3 of Appendix.

If we sum up all the pairing values listed in each column of TABULAR 3, then we get the paring values of $(*)$:

$$
\begin{aligned}
& ay^2(1+3x+5x^2+6x^3+5x^4+3x^5+x^6) \\
&\quad -ay(s^3+rs^2+r^2s+r^3)\cdot(1+3x+4x^2+3x^3+x^4)\cdot(1+\bar{x}yx^2) \\
&\quad +ars(s^4+rs^3+2r^2s^2+r^3s+r^4)\cdot[\,1+2x+x^2 \\
&\quad +\bar{x}xy(1+4x+6x^2+4x^3+x^4)+\bar{x}^2y^2x^4\cdot(1+2x+x^2)\,] \\
&\quad -a(rs)^3(s^3+rs^2+r^2s+r^3)\cdot(1+3x+4x^2+3x^3+x^4)\cdot(\bar{x}+\bar{x}^2x^2y) \\
&\quad +(rs)^6a\bar{x}^2(1+3x+5x^2+6x^3+5x^4+3x^5+x^6) \\
&= 2ar^{-1}s^{-1}\cdot rs\cdot(s^4+3rs^3+4r^2s^2+3r^3s+r^4)\cdot(s^2+r^2) \\
&\quad +2ar^{-1}s^{-1}\cdot(s^4+3rs^3+4r^2s^2+3r^3s+r^4)\cdot(s^2+rs+r^2)\cdot(s^2+r^2) \\
&\quad -2ar^{-1}s^{-1}\cdot(s^4+3rs^3+4r^2s^2+3r^3s+r^4)\cdot(s^2+r^2)\cdot(r+s)^2 \\
&= 2ar^{-1}s^{-1}\cdot(s^4+3rs^3+4r^2s^2+3r^3s+r^4)\cdot(s^2+r^2)\cdot(r+s)^2 \\
&\quad -2ar^{-1}s^{-1}\cdot(s^4+3rs^3+4r^2s^2+3r^3s+r^4)\cdot(s^2+r^2)\cdot(r+s)^2 \\
&= 0 \qquad (\text{because } x=\langle\omega_1',\,\omega_1\rangle=rs^{-1},\ \bar{x}=\langle\omega_1',\,\omega_2\rangle=r^{-3},\ y=\langle\omega_2',\,\omega_1\rangle=s^3).
\end{aligned}
$$

(iv) $X=f_1^3f_2f_1$. By calculation, we get 120 relevant terms of $\triangle^{(4)}(X)$ in $(*)$ and their pairing values listed in TABULAR 4 of Appendix.

If we sum up all the pairing values listed in each column of TABULAR 4, then we get the paring-values of $(*)$:

$$\begin{aligned}
&ay(1{+}3x{+}5x^2{+}6x^3{+}5x^4{+}3x^5{+}x^6) - a(r{+}s)(r^2{+}s^2)[\,1{+}2x{+}2x^2{+}x^3\\
&\quad + x\bar{x}y(1{+}3x{+}5x^2{+}5x^3{+}3x^4{+}x^5)\,] + ars(r^2{+}s^2)(r^2{+}rs{+}s^2)\cdot\\
&\quad [\,\bar{x}(1{+}3x{+}4x^2{+}3x^3{+}x^4) + \bar{x}^2yx^2(1{+}3x{+}4x^2{+}3x^3{+}x^4)\,]\\
&\quad - a(rs)^3(r{+}s)(r^2{+}s^2)[\,\bar{x}^2(1{+}3x{+}5x^2{+}5x^3{+}3x^4{+}x^5)\\
&\quad + \bar{x}^3y(x^3{+}2x^4{+}2x^5{+}x^6)\,] + ar^6s^6\bar{x}^3(1{+}3x{+}5x^2{+}6x^3{+}5x^4{+}3x^5{+}x^6)\\
&= as^{-3}(r{+}s)^2(r^2{+}s^2)(r^2{+}rs{+}s^2)\\
&\quad - as^{-3}r^{-3}(r{+}s)^2(r^2{+}s^2)(r^2{+}rs{+}s^2)(r^2s{+}2r^3{+}rs^2)\\
&\quad + as^{-3}r^{-3}(r^2{+}s^2)(r^2{+}rs{+}s^2)^2(r{+}s)^3 - ar^{-3}s^{-2}(r{+}s)^2(r^2{+}s^2)(r^2{+}rs{+}2s^2)\\
&\quad + ar^{-3}(r{+}s)^2(r^2{+}s^2)(r^2{+}rs{+}s^2)\\
&= a(r{+}s)^2(r^2{+}s^2)(r^2{+}rs{+}s^2)[s^{-3} - s^{-3}r^{-3}(r^2s{+}rs^2{+}2r^3)\\
&\quad + r^{-3}s^{-3}(r^2{+}rs{+}s^2)(r{+}s) - r^{-3}s^{-2}(r^2{+}rs{+}2s^2) + r^{-3}]\\
&= 0.
\end{aligned}$$

(v) $X = f_1f_2f_1^3$. By calculation, we get 120 relevant terms of $\triangle^{(4)}(X)$ in $(*)$ and their pairing values of $(*)$ listed as in TABULAR 5 of Appendix.

If we sum up all the pairing-values in TABULAR 5, then we get the paring value of $(*)$:

$$\begin{aligned}
&ay^3(1{+}3x{+}5x^2{+}6x^3{+}5x^4{+}3x^5{+}x^6) - a(r{+}s)(r^2{+}s^2)[\,\bar{x}y^3x^3(1{+}2x\\
&\quad {+}2x^2{+}x^3){+}y^2(1{+}3x{+}5x^2{+}5x^3{+}3x^4{+}x^5)\,] + ars(r^2{+}s^2)(r^2{+}rs{+}s^2)\cdot\\
&\qquad [\,y(1{+}3x{+}4x^2{+}3x^3{+}x^4) + \bar{x}y^2x^2(1{+}3x{+}4x^2{+}3x^3{+}x^4)\,]\\
&\quad - a(rs)^3(r{+}s)(r^2{+}s^2)[\,\bar{x}yx(1{+}3x{+}5x^2{+}5x^3{+}3x^4{+}x^5){+}1{+}2x{+}2x^2{+}x^3\,]\\
&\quad + ar^6s^6\bar{x}(1{+}3x{+}5x^2{+}6x^3{+}5x^4{+}3x^5{+}x^6)\\
&= a(r{+}s)^3(r^2{+}s^2)(r^2{+}rs{+}s^2) - as(r{+}s)^2(r^2{+}s^2)(r^2{+}rs{+}s^2)(r^2{+}rs{+}2s^2)\\
&\quad + a(r^2{+}s^2)(r^2{+}rs{+}s^2)^2(r{+}s)^3 - ar(r{+}s)^2(r^2{+}s^2)(r^2{+}rs{+}s^2)(2r^2{+}rs{+}s^2)\\
&= a(r{+}s)^2(r^2{+}s^2)(r^2{+}rs{+}s^2)[\,r^3{+}s^3{-}s(r^2{+}rs{+}2s^2)\\
&\quad {+}(r{+}s)(r^2{+}rs{+}s^2){-}r(2r^2{+}rs{+}s^2)\,]\\
&= 0.
\end{aligned}$$

Up to now, these five cases of $\Delta^{(4)}(X)$ have been checked. The proof is completed by checking that the relations in B'^{cop} are preserved for G_2. □

DEFINITION 2.4. For any two Hopf algebras $\mathcal{A}$ and $\mathcal{U}$ connected by a skew-dual pairing $\langle\,,\rangle$ one may form the Drinfel'd quantum double $\mathcal{D}(\mathcal{A},\mathcal{U})$ as in $[KS, 3.2]$, which is a Hopf algebra whose underlying coalgebra is $\mathcal{A}\otimes\mathcal{U}$ with the tensor product coalgebra structure, whose algebra structure is defined by

$$(a\otimes f)(a'\otimes f') = \sum\langle\mathcal{S}_{\mathcal{U}}(f_{(1)}),\, a'_{(1)}\rangle\langle(f_{(3)}),\, a'_{(3)}\rangle aa'_{(2)}\otimes f_{(2)}f', \tag{6}$$

for $a,\ a'\in\mathcal{A}$ and $f,\ f'\in\mathcal{U}$, and whose antipode S is given by

$$S(a\otimes f) = (1\otimes\mathcal{S}_{\mathcal{U}}(f))(\mathcal{S}_{\mathcal{A}}(a)\otimes 1). \tag{7}$$

Clearly, both mappings $\mathcal{A} \ni a \mapsto a\otimes 1 \in \mathcal{D}(\mathcal{A}, \mathcal{U})$ and $\mathcal{U} \ni f \mapsto 1\otimes f \in \mathcal{D}(\mathcal{A}, \mathcal{U})$ are injective Hopf algebra homomorphisms. Denote the image $a\otimes 1$ of a in $\mathcal{D}(\mathcal{A}, \mathcal{U})$ by $\hat{a}$ and the image $1\otimes f$ of f by $\hat{f}$. By (6), we have the following cross relations between elements $\hat{a}$ (for $a \in \mathcal{A}$) and $\hat{f}$ (for $f \in \mathcal{U}$) in the algebra $\mathcal{D}(\mathcal{A}, \mathcal{U})$:

$$\hat{f}\hat{a} = \sum \langle \mathcal{S}_{\mathcal{U}}(f_{(1)}),\, a_{(1)}\rangle \langle (f_{(3)}),\, a_{(3)}\rangle \hat{a}_{(2)}\hat{f}_{(2)}, \tag{8}$$

$$\sum \langle f_{(1)},\, a_{(1)}\rangle \hat{f}_{(2)}\hat{a}_{(2)} = \sum \hat{a}_{(1)}\hat{f}_{(1)}\langle f_{(2)},\, a_{(2)}\rangle. \tag{9}$$

In fact, as an algebra the double $\mathcal{D}(\mathcal{A}, \mathcal{U})$ is the universal algebra generated by the algebras $\mathcal{A}$ and $\mathcal{U}$ with cross relations (8) or, equivalently, (9).

Theorem 2.5. *The two-parameter quantum group $U_{r,\,s}(G_2)$ is isomorphic to the Drinfel'd quantum double $\mathcal{D}(\mathcal{B}, \mathcal{B}')$.*

The proof is the same as that of [BGH1, Theorem 2.5].

Remark 2.6. The proofs of Proposition 2.3 and Theorem 2.5 show the compatibility of the defining relations of $U_{r,s}(G_2)$, where the proof of Theorem 2.5 indicates that the cross relations between $\mathcal{B}$ and $\mathcal{B}'$ are precisely half the ones appearing in $(G1)$–$(G4)$, and the proof of Proposition 2.3 then shows the compatibility of the remaining relations appearing in $\mathcal{B}$ and $\mathcal{B}'$ including the other half of $(G1)$–$(G4)$ and the (r,s)-Serre relations $(G5)$–$(G6)$.

3. Lusztig's Symmetries from $U_{r,\,s}(G_2)$ to $U_{s^{-1},\,r^{-1}}(G_2)$

As we did in [BGH1] for the classical types A, B, C, D, we call $(U_{s^{-1},r^{-1}}(G_2), \langle\,|\,\rangle)$ the associated quantum group corresponding to $(U_{r,s}(G_2), \langle\,,\rangle)$, where the pairing $\langle \omega_i' |\, \omega_j\rangle$ is defined by replacing (r,s) with (s^{-1}, r^{-1}) in the defining formula for $\langle \omega_i', \omega_j\rangle$. Obviously,

$$\langle \omega_i' |\, \omega_j\rangle = \langle \omega_j', \omega_i\rangle.$$

We now study Lusztig's symmetry property between $(U_{r,s}(G_2), \langle\,,\rangle)$ and its associated object $(U_{s^{-1},r^{-1}}(G_2), \langle\,|\,\rangle)$, which indeed indicates the difference in structures between the two-parameter quantum group introduced above and the usual one-parameter quantum group of Drinfel'd-Jimbo type.

To define the Lusztig's symmetries, we introduce the notation of divided-power elements (in $(U_{s^{-1},r^{-1}}(G_2), \langle\,|\,\rangle)$). For any nonnegative integer $k \in \mathbb{N}$, set

$$\langle k\rangle_i = \frac{s_i^{-k} - r_i^{-k}}{s_i^{-1} - r_i^{-1}}, \qquad \langle k\rangle_i! = \langle 1\rangle_i \langle 2\rangle_i \cdots \langle k\rangle_i,$$

and for any element $e_i, f_i \in (U_{s^{-1},r^{-1}}(G_2), \langle\,|\,\rangle)$, define the divided-power elements

$$e_i^{(k)} = e_i^k / \langle k\rangle_i!\,, \qquad f_i^{(k)} = f_i^k / \langle k\rangle_i!\,.$$

Definition 3.1. To every i $(i = 1, 2)$, there corresponds a $\mathbb{Q}$-linear mapping $\mathcal{T}_i :\ (U_{r,s}(G_2), \langle\,,\rangle) \longrightarrow (U_{s^{-1},\,r^{-1}}(G_2), \langle\,|\,\rangle)$ such that $\mathcal{T}_i(r) = s^{-1}$, $\mathcal{T}_i(s) = r^{-1}$, which acts on the generators ω_j, ω_j', e_j, f_j $(1 \le j \le 2)$ as

$$\mathcal{T}_i(\omega_j)=\omega_j\,\omega_i^{-a_{ij}},\qquad \mathcal{T}_i(\omega_j')=\omega_j'\,\omega_i'^{\,-a_{ij}},$$
$$\mathcal{T}_i(e_i)=-\,\omega_i'^{\,-1}f_i,\qquad \mathcal{T}_i(f_i)=-(r_is_i)\,e_i\,\omega_i^{-1},$$

and for $i\neq j$,

$$\mathcal{T}_i(e_j)=\sum_{\nu=0}^{-a_{ij}}(-1)^{\nu}(rs)^{\frac{\nu}{2}(-a_{ij}-\nu)}\langle\omega_j',\omega_i\rangle^{-\nu}\langle\omega_i',\omega_i\rangle^{\frac{\nu}{2}(1+a_{ij})}e_i^{(\nu)}e_j\,e_i^{(-a_{ij}-\nu)},$$
$$\mathcal{T}_i(f_j)=(r_js_j)^{\delta_{ij}^{+}}\sum_{\nu=0}^{-a_{ij}}(-1)^{\nu}(rs)^{\frac{\nu}{2}(-a_{ij}-\nu)}\langle\omega_i',\omega_j\rangle^{\nu}\langle\omega_i',\omega_i\rangle^{-\frac{\nu}{2}(1+a_{ij})}f_i^{(-a_{ij}-\nu)}f_j\,f_i^{(\nu)},$$

here (a_{ij}) is the Cartan matrix of the simple Lie algebra $\mathfrak{g}$ of type G_2, and for any $i\neq j$,

$$\delta_{ij}^{+}=\begin{cases}2, & \text{if }\ i<j,\ \&\ a_{ij}\neq 0,\\ 1, & \text{otherwise .}\end{cases}$$

Lemma 3.2. *$\mathcal{T}_i\,(i=1,2)$ preserves the defining relations $(G1)$–$(G3)$ of $U_{r,s}(G_2)$ into its associated object $U_{s^{-1},r^{-1}}(G_2)$.*

Proof. For G_2, we have

$$\langle\omega_1',\,\omega_1\rangle=rs^{-1}=\langle\omega_1'|\,\omega_1\rangle,\qquad \langle\omega_1',\,\omega_2\rangle=r^{-3}=\langle\omega_2'|\,\omega_1\rangle,$$
$$\langle\omega_2',\,\omega_1\rangle=s^{3}=\langle\omega_1'|\,\omega_2\rangle,\qquad \langle\omega_2',\,\omega_2\rangle=r^{3}s^{-3}=\langle\omega_2'|\,\omega_2\rangle.$$

We show that $\mathcal{T}_1,\mathcal{T}_2$ preserve the defining relations $(G1)$–$(G3)$. $(G1)$ are automatically satisfied. To check $(G2)$ and $(G3)$: first of all, by direct calculation, we have $\mathcal{T}_k(\langle\omega_i',\,\omega_j\rangle)=\langle\mathcal{T}_k(\omega_i'),\,\mathcal{T}_k(\omega_j)\rangle=\langle\omega_j',\,\omega_i\rangle=\langle\omega_i'|\,\omega_j\rangle$, for $i,j,k\in\{1,\,2\}$. This fact ensures that $\mathcal{T}_k(k=1,2)$ preserve $(G2)$ and $(G3)$, that is,

$$\mathcal{T}_k(\omega_j)\mathcal{T}_k(e_i)\mathcal{T}_k(\omega_j)^{-1}=\langle\omega_i'|\,\omega_j\rangle\mathcal{T}_k(e_i),\quad \mathcal{T}_k(\omega_j)\mathcal{T}_k(f_i)\mathcal{T}_k(\omega_j)^{-1}=\langle\omega_i'|\,\omega_j\rangle^{-1}\mathcal{T}_k(f_i),$$
$$\mathcal{T}_k(\omega_j')\mathcal{T}_k(e_i)\mathcal{T}_k(\omega_j')^{-1}=\langle\omega_j'|\,\omega_i\rangle^{-1}\mathcal{T}_k(e_i),\quad \mathcal{T}_k(\omega_j')\mathcal{T}_k(f_i)\mathcal{T}_k(\omega_j')^{-1}=\langle\omega_j'|\,\omega_i\rangle\mathcal{T}_k(f_i),$$

where checking other three identities is equivalent to checking the first one. □

Lemma 3.3. *$\mathcal{T}_i\,(i=1,2)$ preserves the defining relations $(G4)$ into its associated object $U_{s^{-1},r^{-1}}(G_2)$.*

Proof. Put $\Delta=r^2+rs+s^2$. To check $(G4)$: for $i=1,2$, we have

$$[\mathcal{T}_i(e_i),\,\mathcal{T}_i(f_i)]=(r_is_i)\omega_i'^{\,-1}(f_ie_i-e_if_i)\omega_i^{-1}=\mathcal{T}_i([e_i,f_i]),$$

$$\begin{aligned}[\mathcal{T}_2(e_1),\mathcal{T}_2(f_1)]&=[e_1e_2-r^3e_2e_1,\,rs(f_2f_1-s^3f_1f_2)]\\&=rs\{f_2[e_1,f_1]e_2+e_1[e_2,f_2]f_1-r^3([e_2,f_2]f_1e_1+e_2f_2[e_1,f_1])\\&\quad-s^3([e_1,f_1]f_2e_2+e_1f_1[e_2,f_2])+(rs)^3(e_2[e_1,f_1]f_2+f_1[e_2,\,f_2]e_1)\}\\&=\frac{\omega_2\omega_1-\omega_2'\omega_1'}{s^{-1}-r^{-1}}=\frac{\mathcal{T}_2(\omega_1)-\mathcal{T}_2(\omega_1')}{s^{-1}-r^{-1}}=\mathcal{T}_2([e_1,f_1]),\end{aligned}$$

and as for

$$
\begin{aligned}
[\mathcal{T}_1(e_2), \mathcal{T}_1(f_2)] = \frac{r^3s^3}{(r+s)^2\Delta^2}\big[\,&(rs^2)^3e_2e_1^3 - rs^3\Delta e_1e_2e_1^2 + s\Delta e_1^2e_2e_1 - e_1^3e_2,\\
&(r^2s)^3f_1^3f_2 - sr^3\Delta f_1^2f_2f_1 + r\Delta f_1f_2f_1^2 - f_2f_1^3\,\big],
\end{aligned}
$$

we have to show that the above bracket on the right-hand side is equal to

$$
\Delta(r+s)^2\cdot\frac{\omega_2\omega_1^3-\omega_2'\omega_1'^3}{r-s}.
$$

To do so, we introduce the notations of "quantum root vectors" in terms of adjoint actions, as follows:

$$
\begin{aligned}
E_{12} &= (\mathrm{ad}_l e_1)(e_2) = e_1e_2 - s^3e_2e_1,\\
F_{12} &= (\mathrm{ad}_r f_1)(f_2) = f_2f_1 - r^3f_1f_2,\\
E_{112} &= (\mathrm{ad}_l e_1)^2(e_2) = e_1E_{12} - rs^2E_{12}e_1,\\
F_{112} &= (\mathrm{ad}_r f_1)^2(f_2) = F_{12}f_1 - r^2sf_1F_{12},\\
E_{1112} &= (\mathrm{ad}_l e_1)^3(e_2) = e_1^3e_2 - s\Delta e_1^2e_2e_1 + rs^3\Delta e_1e_2e_1^2 - (rs^2)^3e_2e_1^3,\\
F_{1112} &= (\mathrm{ad}_r f_1)^3(f_2) = f_2f_1^3 - r\Delta f_1f_2f_1^2 + sr^3\Delta f_1^2f_2f_1 - (r^2s)^3f_1^3f_2.
\end{aligned}
$$

That is, we need to verify that

$$
[\,E_{1112}, F_{1112}\,] = \Delta(r+s)^2\cdot\frac{\omega_2\omega_1^3-\omega_2'\omega_1'^3}{r-s}.
$$

By direct calculation using the Leibniz rule, we have

$$
\begin{gathered}
[e_1, F_{12}] = -\Delta\omega_1f_2, \qquad [e_2, F_{12}] = f_1\omega_2',\\
[E_{12}, f_1] = -\Delta e_2\omega_1', \qquad [E_{12}, f_2] = \omega_2e_1,\\
[E_{12}, F_{12}] = \frac{\omega_1\omega_2-\omega_1'\omega_2'}{r-s},\\
[e_1, F_{112}] = -(r+s)^2\omega_1F_{12}, \qquad [e_2, F_{112}] = s(s^2-r^2)f_1^2\omega_2',\\
[E_{112}, f_1] = -(r+s)^2E_{12}\omega_1', \qquad [E_{112}, f_2] = r(r^2-s^2)\omega_2e_1^2,\\
[E_{112}, F_{12}] = (r+s)^2\omega_1\omega_2e_1, \qquad [E_{12}, F_{112}] = (r+s)^2f_1\omega_1'\omega_2',\\
[E_{112}, F_{112}] = (r+s)^2\cdot\frac{\omega_1^2\omega_2-\omega_1'^2\omega_2'}{r-s},
\end{gathered}
$$

as well as

$$
\begin{aligned}
[e_1, F_{1112}] &= [e_1, F_{112}f_1 - rs^2f_1F_{112}] = -\Delta\omega_1F_{112},\\
[E_{112}, F_{1112}] &= [E_{112}, F_{112}f_1 - rs^2f_1F_{112}]\\
&= [E_{112}, F_{112}]f_1 - rs^2f_1[E_{112}, F_{112}] + F_{112}[E_{112}, f_1] - rs^2[E_{112}, f_1]F_{112}\\
&= \Delta(r+s)^2f_1\omega_1'^2\omega_2',
\end{aligned}
$$

$$\begin{aligned}
[E_{1112}, F_{1112}] &= [e_1E_{112} - r^2sE_{112}e_1, F_{1112}] \\
&= [e_1, F_{1112}]E_{112} - r^2sE_{112}[e_1, F_{1112}] + e_1[E_{112}, F_{1112}] - r^2s[E_{112}, F_{1112}]e_1 \\
&= \Delta\omega_1[E_{112}, F_{112}] + \Delta(r+s)^2[e_1, f_1]\omega_1'^2\omega_2' \\
&= \Delta(r+s)^2 \cdot \frac{\omega_2\omega_1^3 - \omega_2'\omega_1'^3}{r-s}.
\end{aligned}$$

Thus, we arrive at $[\mathcal{T}_1(e_2), \mathcal{T}_1(f_2)] = \mathcal{T}_1([e_2,\, f_2]) \in U_{s^{-1},\,r^{-1}}(G_2)$. □

Lemma 3.4. *$\mathcal{T}_2$ preserves the (r,s)-Serre relations $(G5)_1, (G6)_1$ into its associated object $U_{s^{-1},r^{-1}}(G_2)$:*

(1) $\mathcal{T}_2(e_2)^2\mathcal{T}_2(e_1) - (r^3+s^3)\mathcal{T}_2(e_2)\mathcal{T}_2(e_1)\mathcal{T}_2(e_2) + (rs)^3\mathcal{T}_2(e_1)\mathcal{T}_2(e_2)^2 = 0,$

(2) $\mathcal{T}_2(f_1)\mathcal{T}_2(f_2)^2 - (r^3+s^3)\mathcal{T}_2(f_2)\mathcal{T}_2(f_1)\mathcal{T}_2(f_2) + (rs)^3\mathcal{T}_2(f_1)\mathcal{T}_2(f_2)^2 = 0.$

Proof. For the degree 2 (r,s)-Serre relation $(G5)_1$

$$e_2^2e_1 - (r^{-3}+s^{-3})e_2e_1e_2 + r^{-3}s^{-3}e_1e_2^2 = 0,$$

observe that

(3) $\mathcal{T}_2(e_1)\mathcal{T}_2(e_2) = r^{-3}\mathcal{T}_2(e_2)\mathcal{T}_2(e_1) - r^{-3}e_1, \qquad \mathcal{T}_2(e_2)e_1 = s^3e_1\mathcal{T}_2(e_2).$

Making $\mathcal{T}_2$ act algebraically on the left-hand side of $(G5)_1$, we have

$$\begin{aligned}
&\mathcal{T}_2(e_2)^2\mathcal{T}_2(e_1) - (r^3+s^3)\mathcal{T}_2(e_2)\mathcal{T}_2(e_1)\mathcal{T}_2(e_2) + (rs)^3\mathcal{T}_2(e_1)\mathcal{T}_2(e_2)^2 \\
&\qquad = \mathcal{T}_2(e_2)r^3(\mathcal{T}_2(e_1)\mathcal{T}_2(e_2) + r^{-3}e_1) - (r^3+s^3)\mathcal{T}_2(e_2)\mathcal{T}_2(e_1)\mathcal{T}_2(e_2) \\
&\qquad\quad + (rs)^3(r^{-3}\mathcal{T}_2(e_2)\mathcal{T}_2(e_1) - r^{-3}e_1)\mathcal{T}_2(e_2) \\
&\qquad = 0,
\end{aligned}$$

proving (1). The proof of (2) is similar. □

To prove that $\mathcal{T}_1$ preserves the Serre relations, we need three auxiliary lemmas.

Lemma 3.5. *In the notation in Lemma 3.3, we have*

$$[E_{1112}E_{112} - r^3E_{112}E_{1112},\, f_2] = 0.$$

Proof. Since $e_1E_{1112} - r^3E_{1112}e_1 = \mathrm{ad}_l(e_1)^4(e_2) = 0$ (Serre relation), and

$$\begin{aligned}
[E_{1112},\, f_2] &= [e_1E_{112} - r^2sE_{112}e_1,\, f_2] = e_1[E_{112},\, f_2] - r^2s[E_{112},\, f_2]e_1 \\
&= r^3(r-s)(r^2-s^2)\,\omega_2e_1^3,
\end{aligned}$$

we obtain

$$\begin{aligned}
&[E_{1112}E_{112} - r^3E_{112}E_{1112},\, f_2] = E_{1112}[E_{112},\, f_2] + [E_{1112},\, f_2]E_{112} \\
&\qquad - r^3\big(E_{112}[E_{1112},\, f_2] + [E_{112},\, f_2]E_{1112}\big) \\
&\quad = r^3(r-s)(r^2-s^2)\,\omega_2\,\big(e_1^3E_{112} - r\Delta e_1E_{1112}e_1 - (r^2s)^3E_{112}e_1^3\big) \\
&\quad = r^3(r-s)(r^2-s^2)\,\omega_2\,\big(e_1^3E_{112} - r\Delta e_1^2E_{112}e_1 + r^3s\Delta e_1E_{112}e_1^2 - (r^2s)^3E_{112}e_1^3\big) \\
&\quad = r^3(r-s)(r^2-s^2)\,\omega_2\,\big(e_1\cdot(\mathcal{SR}) - rs^2(\mathcal{SR})\cdot e_1\big) \\
&\quad = 0,
\end{aligned}$$

where $(\mathcal{SR})$ denotes the left-hand-side presentation of the (r,s)-Serre relation $(G5)_2$

$$e_1^2E_{112} - r^2(r+s)e_1E_{112}e_1 + r^5sE_{112}e_1^2 = 0,$$

and we used the replacement $E_{1112} = e_1E_{112} - r^2sE_{112}e_1$ in the third equality. □

LEMMA 3.6. *In the notation in Lemma 3.3, we have*

$$[E_{1112}E_{112} - r^3E_{112}E_{1112},\, f_1] = 0.$$

PROOF. It is easy to check that $[E_{1112}, f_1] = -\Delta E_{112}\omega_1'$. Thus

$$\begin{aligned}
[E_{1112}E_{112} - r^3E_{112}E_{1112},\, f_1] &= E_{1112}[E_{112},\, f_1] + [E_{1112},\, f_1]E_{112} \\
&\quad - r^3\big(E_{112}[E_{1112},\, f_1] + [E_{112},\, f_1]E_{1112}\big) \\
&= (r+s)\left[(r+s)((rs)^3E_{12}E_{1112} - E_{1112}E_{12}) + r(r-s)\Delta E_{112}^2\right]\omega_1'.
\end{aligned}$$

It suffices to show that

$$E_{1112}E_{12} = (rs)^3E_{12}E_{1112} + r(r-s)(r+s)^{-1}\Delta E_{112}^2. \tag{4}$$

At first, we note that the (r,s)-Serre relation $(G5)_1$ is equivalent to

$$E_{12}e_2 = r^3e_2E_{12}.$$

As $e_1e_2 = E_{12} + s^3e_2e_1$, we get

$$\begin{aligned}
E_{112}e_2 &= (e_1E_{12} - rs^2E_{12}e_1)e_2 = r^3e_1e_2E_{12} - rs^2E_{12}e_1e_2 \\
&= r^3(E_{12} + s^3e_2e_1)E_{12} - rs^2E_{12}(E_{12} + s^3e_2e_1) \\
&= r(r^2-s^2)E_{12}^2 + (rs)^3e_2(e_1E_{12} - rs^2E_{12}e_1) \\
&= r(r^2-s^2)E_{12}^2 + (rs)^3e_2E_{112}.
\end{aligned}$$

Next, we claim

$$E_{1112}e_2 = (rs^2)^3e_2E_{1112} - r(rs-r^2+s^2)E_{112}E_{12} + (rs)^2(r^2+rs-s^2)E_{12}E_{112}.$$

Indeed, since $E_{1112} = e_1E_{112} - r^2sE_{112}e_1, E_{112} = e_1E_{12} - rs^2E_{12}e_1,\ e_1e_2 = E_{12} + s^3e_2e_1$, we have

$$\begin{aligned}
E_{1112}e_2 &= e_1(E_{112}e_2) - r^2sE_{112}(e_1e_2) \\
&= r(r^2-s^2)e_1E_{12}^2 + (rs)^3(e_1e_2)E_{112} - r^2sE_{112}(e_1e_2) \\
&= r(r^2-s^2)e_1E_{12}^2 + (rs)^3E_{12}E_{112} + (rs^2)^3e_2e_1E_{112} - r^2sE_{112}E_{12} \\
&\quad - (rs^2)^2(E_{112}e_2)e_1 \\
&= r(r^2-s^2)E_{112}E_{12} + (rs)^2(r^2-s^2)E_{12}e_1E_{12} + (rs)^3E_{12}E_{112} \\
&\quad + (rs^2)^3e_2e_1E_{112} - r^2sE_{112}E_{12} - r^3s^4(r^2-s^2)E_{12}^2e_1 - r^5s^7e_2E_{112}e_1 \\
&= (rs^2)^3e_2E_{1112} - r(rs-r^2+s^2)E_{112}E_{12} + (rs)^2(r^2+rs-s^2)E_{12}E_{112}.
\end{aligned}$$

To prove (4), we first note that

$$\begin{aligned}
&[(r+s)((rs)^3E_{12}E_{1112} - E_{1112}E_{12}) + r(r-s)\Delta E_{112}^2,\, f_1] \\
&= (r+s)(rs)^3\big(E_{12}[E_{1112},\, f_1] + [E_{12},\, f_1]E_{1112}\big) \\
&\quad - (r+s)\big(E_{1112}[E_{12},\, f_1] + [E_{1112},\, f_1]E_{12}\big) \\
&\quad + r(r-s)\Delta(E_{112}[E_{112}, f_1] + [E_{112},\, f_1]E_{112}) \\
&= -(r+s)(rs)^3\Delta\big(E_{12}E_{112} + s^3e_2E_{1112}\big)\omega_1' \\
&\quad + (r+s)\Delta\big(E_{1112}e_2 + r^2sE_{112}E_{12}\big)\omega_1' \\
&\quad - r(r-s)(r+s)^2\Delta\big(E_{112}E_{12} + rs^2E_{12}E_{112}\big)\omega_1',
\end{aligned}$$

which vanishes by the preceding identity. Second, instead of f_1 by f_2 in the above formula, we get

$$\begin{aligned}
&[(r+s)((rs)^3E_{12}E_{1112}-E_{1112}E_{12})+r(r-s)\Delta E_{112}^2,\ f_2]\\
&\quad=(r+s)(rs)^3\big(r^3(r^2-s^2)(r-s)E_{12}\omega_2e_1^3+\omega_2e_1E_{1112}\big)\\
&\qquad-(r+s)\big(E_{1112}\omega_2e_1+r^3(r^2-s^2)(r-s)\omega_2e_1^3E_{12}\big)\\
&\qquad+r^2(r-s)(r^2-s^2)\Delta\big(E_{112}\omega_2e_1^2+\omega_2e_1^2E_{112}\big)\\
&\quad=(r+s)(rs)^3\omega_2\big((rs)^3(r^2-s^2)(r-s)E_{12}e_1^3+e_1E_{1112}\big)\\
&\qquad-(r+s)\omega_2\big((r^2s)^3E_{1112}e_1+r^3(r^2-s^2)(r-s)e_1^3E_{12}\big)\\
&\qquad+r^2(r-s)(r^2-s^2)\Delta\omega_2\big((rs)^3E_{112}e_1^2+e_1^2E_{112}\big)\\
&\quad=r^2(r-s)(r^2-s^2)\Delta\omega_2\big((rs)^3E_{112}e_1^2+e_1^2E_{112}\big)\\
&\qquad-r^3(r^2-s^2)^2\omega_2\big(e_1^3E_{12}-(rs^2)^3E_{12}e_1^3\big)\\
&\quad=r^2(r-s)(r^2-s^2)\Delta\omega_2\big((rs)^3E_{112}e_1^2+e_1^2E_{112}\big)\\
&\qquad-r^3(r^2-s^2)^2\omega_2\big(e_1^2E_{112}+rs^2e_1E_{112}e_1+(rs^2)^2E_{112}e_1^2\big)\\
&\quad=(rs)^2(r-s)(r^2-s^2)\omega_2\big(e_1^2E_{112}-r^2(r+s)e_1E_{112}e_1+r^5sE_{112}e_1^2\big)\\
&\quad=(rs)^2(r-s)(r^2-s^2)\omega_2(\mathrm{ad}_le_1)^2(E_{112})\\
&\quad=(rs)^2(r-s)(r^2-s^2)\omega_2(\mathrm{ad}_le_1)^4(e_2)\\
&\quad=0.
\end{aligned}$$

Then, through an argument similar to the one used in the deduction of [BKL, Lemma 3.4], we get (4). □

By [BKL, Lemma 3.4], Lemmas 3.5 and 3.6 imply:

LEMMA 3.7.

$$E_{1112}E_{112}-r^3E_{112}E_{1112}=0.$$

LEMMA 3.8. *$\mathcal{T}_1$ preserves the (r,s)-Serre relations $(G5)_1,(G6)_1$ into its associated object $U_{s^{-1},r^{-1}}(G_2)$, i.e.,*

$$\mathcal{T}_1(e_2)^2\mathcal{T}_1(e_1)-(r^3+s^3)\mathcal{T}_1(e_2)\mathcal{T}_1(e_1)\mathcal{T}_1(e_2)+(rs)^3\mathcal{T}_1(e_1)\mathcal{T}_1(e_2)^2=0, \tag{5}$$

$$\mathcal{T}_1(f_1)\mathcal{T}_1(f_2)^2-(r^3+s^3)\mathcal{T}_1(f_2)\mathcal{T}_1(f_1)\mathcal{T}_1(f_2)+(rs)^3\mathcal{T}_1(f_2)^2\mathcal{T}_1(f_1)=0. \tag{6}$$

PROOF. By direct calculation, we have

$$\begin{aligned}
\mathcal{T}_1(e_2)\mathcal{T}_1(e_1)&=\Big[-\frac{1}{s^3(r+s)\Delta}E_{1112}\Big]\cdot(-\omega_1'^{-1}f_1)\\
&=s^3\mathcal{T}_1(e_1)\mathcal{T}_1(e_2)-\frac{1}{rs^2(r+s)}E_{112}.
\end{aligned}\tag{7}$$

Hence, to prove (5) is equivalent to prove

$$\mathcal{T}_1(e_2)E_{112}-r^3E_{112}\mathcal{T}_1(e_2)=0.$$

However, the latter is given by Lemma 3.7.

The proof of (6) is analogous. □

To prove that $\mathcal{T}_2$ preserves the Serre relations, we also need auxiliary lemmas. Write

$$E_{21}:=(ad_le_2)(e_1)=e_2e_1-r^{-3}e_1e_2,$$

and note that $(G5)_1$ is equivalent to $(ad_l e_2)(E_{21}) = e_2E_{21} - s^{-3}E_{21}e_2 = 0$, i.e., $E_{21}e_2 = s^3e_2E_{21}$.

Lemma 3.9.

$$\big[\, e_1E_{21}^3 - s\Delta E_{21}e_1E_{21}^2 + rs^3\Delta E_{21}^2e_1E_{21} - (rs^2)^3E_{21}^3e_1,\, f_1 \,\big] = 0.$$

Proof. Since

$$\begin{aligned} [E_{21},\, f_1] &= r^{-3}\Delta e_2\omega_1, \qquad & \omega_1E_{21} &= rs^2E_{21}\omega_1,\\ [E_{21}^2,\, f_1] &= r^{-3}s^{-1}(r+s)\Delta E_{21}e_2\omega_1, \qquad & \omega_1'E_{21} &= r^2sE_{21}\omega_1', \end{aligned} \tag{8}$$
$$[E_{21}^3,\, f_1] = r^{-3}s^{-2}\Delta^2E_{21}^2e_2\omega_1,$$

we get

$$\begin{aligned} \textstyle\sum_1 &= \frac{\omega_1-\omega_1'}{r-s}E_{21}^3 - (rs^2)^3E_{21}^3\frac{\omega_1-\omega_1'}{r-s} - s\Delta E_{21}\frac{\omega_1-\omega_1'}{r-s}E_{21}^2 + rs^3\Delta E_{21}^2\frac{\omega_1-\omega_1'}{r-s}E_{21}\\ &= -(rs)^3\Delta E_{21}^3\omega_1' + rs^2\Delta E_{21}^2\omega_1'E_{21}\\ &= 0, \end{aligned}$$

and

$$\begin{aligned} &\big[\, e_1E_{21}^3 - s\Delta E_{21}e_1E_{21}^2 + rs^3\Delta E_{21}^2e_1E_{21} - (rs^2)^3E_{21}^3e_1,\, f_1 \,\big]\\ &\quad= e_1[E_{21}^3,\, f_1] + [e_1,\, f_1]E_{21}^3 - (rs^2)^3\big(E_{21}^3[e_1,\, f_1] + [E_{21}^3,\, f_1]e_1\big)\\ &\qquad - s\Delta\big(E_{21}e_1[E_{21}^2,\, f_1] + E_{21}[e_1,\, f_1]E_{21}^2 + [E_{21},\, f_1]e_1E_{21}^2\big)\\ &\qquad + rs^3\Delta\big(E_{21}^2e_1[E_{21},\, f_1] + E_{21}^2[e_1,\, f_1]E_{21} + [E_{21}^2,\, f_1]e_1E_{21}\big)\\ &\quad= r^{-3}s^{-2}\Delta^2e_1E_{21}^2e_2\omega_1 + \frac{\omega_1-\omega_1'}{r-s}E_{21}^3\\ &\qquad - s\Delta\big[r^{-3}s^{-1}(r{+}s)\Delta E_{21}e_1E_{21}e_2\omega_1 + E_{21}\frac{\omega_1-\omega_1'}{r-s}E_{21}^2 + r^{-3}\Delta e_2\omega_1e_1E_{21}^2\big]\\ &\qquad + rs^3\Delta\big[r^{-3}\Delta E_{21}^2e_1e_2\omega_1 + E_{21}^2\frac{\omega_1-\omega_1'}{r-s}E_{21} + r^{-3}s^{-1}(r{+}s)\Delta E_{21}e_2\omega_1e_1E_{21}\big]\\ &\qquad - (rs^2)^3\big(E_{21}^3\frac{\omega_1-\omega_1'}{r-s} + r^{-3}s^{-2}\Delta^2E_{21}^2e_2\omega_1e_1\big)\\ &\quad= \textstyle\sum_1 + (r^{-3}s^{-2}\Delta^2)\sum_2\omega_1 = (r^{-3}s^{-2}\Delta^2)\sum_2\omega_1, \end{aligned}$$

where

$$\begin{aligned} \textstyle\sum_2 &= e_1E_{21}^2e_2 - s^2(r+s)E_{21}e_1E_{21}e_2 - (rs^2)^3e_2e_1E_{21}^2\\ &\qquad + rs^5E_{21}^2e_1e_2 + r^3s^5(r+s)E_{21}e_2e_1E_{21} - r^4s^5E_{21}^2e_2e_1. \end{aligned}$$

We next show $\sum_2 = 0$. As $E_{21}e_2 = s^3e_2E_{21}$ and $e_2e_1 - r^{-3}e_1e_2 = E_{21}$, we get

$$\begin{aligned} \textstyle\sum_2 &= \big(e_1E_{21}^2e_2 - (rs^2)^3e_2e_1E_{21}^2\big) + \big(rs^5E_{21}^2e_1e_2 - r^4s^5E_{21}^2e_2e_1\big)\\ &\quad - s^2(r+s)E_{21}e_1E_{21}e_2 + r^3s^5(r+s)E_{21}e_2e_1E_{21}\\ &= -(rs^2)^3E_{21}^3 - r^4s^5E_{21}^3 + r^3s^5(r+s)E_{21}^3\\ &= 0. \end{aligned}$$

This completes the proof. □

Lemma 3.10.

$$\big[\, e_1E_{21}^3 - s\Delta E_{21}e_1E_{21}^2 + rs^3\Delta E_{21}^2e_1E_{21} - (rs^2)^3E_{21}^3e_1,\, f_2 \,\big] = 0.$$

Proof. Noting that

$$
\begin{aligned}
[E_{21},\, f_2] = -r^{-3}\omega_2' e_1, \qquad\qquad E_{21}\omega_2' = r^3\omega_2' E_{21},\\
[E_{21}^2,\, f_2] = -r^{-3}\omega_2'(e_1E_{21} + r^3E_{21}e_1),\\
[E_{21}^3,\, f_2] = -r^{-3}\omega_2'(e_1E_{21}^2 + r^3E_{21}e_1E_{21} + r^6E_{21}^2e_1),
\end{aligned} \tag{9}
$$

we obtain

$$
\begin{aligned}
&\big[\, e_1E_{21}^3 - s\Delta E_{21}e_1E_{21}^2 + rs^3\Delta E_{21}^2e_1E_{21} - (rs^2)^3E_{21}^3e_1,\ f_2\,\big]\\
&\quad = e_1[E_{21}^3,\, f_2] - s\Delta\big(E_{21}e_1[E_{21}^2,\, f_2] + [E_{21},\, f_2]e_1E_{21}^2\big)\\
&\qquad + rs^3\Delta\big(E_{21}^2e_1[E_{21},\, f_2] + [E_{21}^2,\, f_2]e_1E_{21}\big) - (rs^2)^3[E_{21}^3,\, f_2]e_1\\
&\quad = -r^{-3}\omega_2'\,\big\{s^3e_1\big(e_1E_{21}^2 + r^3E_{21}e_1E_{21} + r^6E_{21}^2e_1\big)\\
&\qquad - s\Delta\big[(rs)^3E_{21}e_1\big(e_1E_{21} + r^3E_{21}e_1\big) + e_1^2E_{21}^2\big]\\
&\qquad + rs^3\Delta\big[(r^2s)^3E_{21}^2e_1^2 + (e_1E_{21} + r^3E_{21}e_1)e_1E_{21}\big]\\
&\qquad -(rs^2)^3(e_1E_{21}^2 + r^3E_{21}e_1E_{21} + r^6E_{21}^2e_1)e_1\big\}\\
&\quad = -r^{-2}s\,\omega_2'\,S.
\end{aligned}
$$

where

$$
\begin{aligned}
S = (rs)^2(r^3{-}s^3)\big(e_1E_{21}^2e_1 + E_{21}e_1^2E_{21}\big) + s^2(2r^2{+}rs{+}s^2)(e_1E_{21})^2\\
- r^5s^3(2s^2{+}rs{+}r^2)(E_{21}e_1)^2 - (r{+}s)\big(e_1^2E_{21}^2 - (rs)^6E_{21}^2e_1^2\big).
\end{aligned}
$$

It remains to prove $S=0$, which by [BKL, Lemma 3.4] is equivalent to showing that $[S, f_1] = 0 = [S, f_2]$. To this end, we first observe:

Lemma 3.11.

$$
e_1^3E_{21} - s\Delta e_1^2E_{21}e_1 + rs^3\Delta e_1E_{21}e_1^2 - (rs^2)^3E_{21}e_1^3 = 0.
$$

Proof. It is easy to see that

$$
e_1^3E_{21} - s\Delta e_1^2E_{21}e_1 + rs^3\Delta e_1E_{21}e_1^2 - (rs^2)^3E_{21}e_1^3 = r^{-3}(\mathrm{ad}_l e_1)^4(e_2),
$$

which is in fact the (r,s)-Serre relation $(G5)_2$ up to a factor r^{-3}. $\square$

Now set $S_i := [S, f_i]$ for $i = 1, 2$. Using (9), we obtain

$$
\begin{aligned}
S_2 &= (rs)^2(r^3-s^3)\big(e_1[E_{21}^2, f_2]e_1 + [E_{21}, f_2]e_1^2E_{21} + E_{21}e_1^2[E_{21}, f_2]\big)\\
&\quad + s^2(2r^2+rs+s^2)\big(e_1[E_{21}, f_2]e_1E_{21} + e_1E_{21}e_1[E_{21}, f_2]\big)\\
&\quad - r^5s^3(2s^2+rs+r^2)\big([E_{21}, f_2]e_1E_{21}e_1 + E_{21}e_1[E_{21}, f_2]e_1\big)\\
&\quad - (r+s)\big(e_1^2[E_{21}^2, f_2] - (rs)^6[E_{21}^2, f_2]e_1^2\big)\\
&= (-r^{-3})\,\big\{(rs)^2(r^3-s^3)\big[e_1\omega_2'(e_1E_{21} + r^3E_{21}e_1)e_1 + \omega_2'e_1^3E_{21} + E_{21}e_1^2\omega_2'e_1\big]\\
&\quad + s^2(2r^2+rs+s^2)\big(e_1\omega_2'e_1^2E_{21} + e_1E_{21}e_1\omega_2'e_1\big)\\
&\quad - r^5s^3(2s^2+rs+r^2)\big(\omega_2'e_1^2E_{21}e_1 + E_{21}e_1\omega_2'e_1^2\big)\\
&\quad - (r+s)\big[e_1^2\omega_2'(e_1E_{21} + r^3E_{21}e_1) - (rs)^6\omega_2'(e_1E_{21} + r^3E_{21}e_1)e_1^2\big]\big\}\\
&= (-r^{-3})\,\omega_2'\,\big\{(rs)^2(r^3-s^3)\big[s^3e_1(e_1E_{21} + r^3E_{21}e_1)e_1 + e_1^3E_{21} + (rs^2)^3E_{21}e_1^3\big]\\
&\quad + s^2(2r^2+rs+s^2)\big(s^3e_1^3E_{21} + (rs^2)^3e_1E_{21}e_1^2\big)\\
&\quad - r^5s^3(2s^2+rs+r^2)\big(e_1^2E_{21}e_1 + (rs)^3E_{21}e_1^3\big)\\
&\quad - (r+s)\big[s^6e_1^2(e_1E_{21} + r^3E_{21}e_1) - (rs)^6(e_1E_{21} + r^3E_{21}e_1)e_1^2\big]\big\}\\
&= (-r^{-3})(rs)^2(r^3+s^3)\,\omega_2'\,\big[\,e_1^3E_{21} - s\Delta e_1^2E_{21}e_1 + rs^3\Delta e_1E_{21}e_1^2 - (rs^2)^3E_{21}e_1^3\,\big]\\
&= 0. \qquad \text{(by Lemma 3.11)}
\end{aligned}
$$

Next we prove that $S_1 = 0$. Using (8) and noting that $[e_1^2,\ f_1] = \frac{r+s}{rs}\cdot\frac{s\omega_1-r\omega_1'}{r-s}e_1$, we can get

$$
\begin{aligned}
S_1 &= (rs)^2(r^3-s^3)\big([e_1, f_1]E_{21}^2e_1 + e_1[E_{21}^2, f_1]e_1 + e_1E_{21}^2[e_1, f_1]\\
&\qquad\qquad + [E_{21}, f_1]e_1^2E_{21} + E_{21}[e_1^2, f_1]E_{21} + E_{21}e_1^2[E_{21}, f_1]\big)\\
&\quad + s^2(2r^2+rs+s^2)\big([e_1, f_1]E_{21}e_1E_{21} + e_1[E_{21}, f_1]e_1E_{21}\\
&\qquad\qquad + e_1E_{21}[e_1, f_1]E_{21} + e_1E_{21}e_1[E_{21}, f_1]\big)\\
&\quad - r^5s^3(2s^2+rs+r^2)\big([E_{21}, f_1]e_1E_{21}e_1 + E_{21}[e_1, f_1]E_{21}e_1\\
&\qquad\qquad + E_{21}e_1[E_{21}, f_1]e_1 + E_{21}e_1E_{21}[e_1, f_1]\big)\\
&\quad - (r+s)\big([e_1^2, f_1]E_{21}^2 + e_1^2[E_{21}^2, f_1] - (rs)^6[E_{21}^2, f_1]e_1^2 - (rs)^6E_{21}^2[e_1^2, f_1]\big)\\
&= (rs)^2(r^3-s^3)\left(\frac{\omega_1-\omega_1'}{r-s}E_{21}^2e_1 + r^{-3}s^{-1}(r+s)\Delta e_1E_{21}e_2\omega_1e_1 + e_1E_{21}^2\frac{\omega_1-\omega_1'}{r-s}\right.\\
&\qquad\qquad \left. + r^{-3}\Delta e_2\omega_1e_1^2E_{21} + \frac{r+s}{rs}E_{21}\frac{s\omega_1-r\omega_1'}{r-s}e_1E_{21} + r^{-3}\Delta E_{21}e_1^2e_2\omega_1\right)\\
&\quad + s^2(2r^2+rs+s^2)\left(\frac{\omega_1-\omega_1'}{r-s}E_{21}e_1E_{21} + r^{-3}\Delta e_1e_2\omega_1e_1E_{21}\right.\\
&\qquad\qquad \left. + e_1E_{21}\frac{\omega_1-\omega_1'}{r-s}E_{21} + r^{-3}\Delta e_1E_{21}e_1e_2\omega_1\right)\\
&\quad - r^5s^3(2s^2+rs+r^2)\left(r^{-3}\Delta e_2\omega_1e_1E_{21}e_1 + E_{21}\frac{\omega_1-\omega_1'}{r-s}E_{21}e_1\right.\\
&\qquad\qquad \left. + r^{-3}\Delta E_{21}e_1e_2\omega_1e_1 + E_{21}e_1E_{21}\frac{\omega_1-\omega_1'}{r-s}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\quad - (r+s)\Big(\frac{r+s}{rs}\frac{s\omega_1-r\omega_1'}{r-s}e_1E_{21}^2 + r^{-3}s^{-1}(r+s)\Delta e_1^2E_{21}e_2\omega_1\\
&\qquad -r^3s^5(r+s)\Delta E_{21}e_2\omega_1e_1^2 - (rs)^5(r+s)E_{21}^2\frac{s\omega_1-r\omega_1'}{r-s}e_1\Big)\\
&= A+B+C+D,
\end{aligned}
$$

where A,B,C,D are given as follows.

Noting that $\omega_1E_{21}=rs^2E_{21}\omega_1$ and $\omega_1'E_{21}=r^2sE_{21}\omega_1'$, we have

$$
\begin{aligned}
A :&= (rs)^2\Delta(\omega_1-\omega_1')E_{21}^2e_1 - r^5s^3(2s^2+rs+r^2)E_{21}\frac{\omega_1-\omega_1'}{r-s}E_{21}e_1\\
&\quad + (rs)^5\frac{(r+s)^2}{r-s}E_{21}^2(s\omega_1-r\omega_1')e_1\\
&= -r^5s^4(r^3-s^3)E_{21}^2e_1\omega_1,\\
B :&= (rs)(r+s)\Delta E_{21}(s\omega_1-r\omega_1')e_1E_{21} + s^2(2r^2+rs+s^2)\frac{\omega_1-\omega_1'}{r-s}E_{21}e_1E_{21}\\
&\quad - r^5s^3(2s^2+rs+r^2)E_{21}e_1E_{21}\frac{\omega_1-\omega_1'}{r-s} = 0,\\
C :&= (rs)^2\Delta e_1E_{21}^2(\omega_1-\omega_1') + s^2(2r^2+rs+s^2)e_1E_{21}\frac{\omega_1-\omega_1'}{r-s}E_{21}\\
&\quad - \frac{(r+s)^2}{rs}\frac{s\omega_1-r\omega_1'}{r-s}e_1E_{21}^2\\
&= rs^2(r^3-s^3)e_1E_{21}^2\omega_1,
\end{aligned}
$$

furthermore, using $E_{21}e_2=s^3e_2E_{21}$, $r^{-3}e_1e_2=e_2e_1-E_{21}$ and $e_2e_1=E_{21}+r^{-3}e_1e_2$, we get

$$
\begin{aligned}
D :&= r^{-3}\Delta\big\{(rs)^2(r^3-s^3)\big[s^{-1}(r+s)e_1E_{21}e_2\omega_1e_1 + e_2\omega_1e_1^2E_{21} + E_{21}e_1^2e_2\omega_1\big]\\
&\quad + s^2(2r^2+rs+s^2)\big[e_1e_2\omega_1e_1E_{21} + e_1E_{21}e_1e_2\omega_1\big]\\
&\quad - r^5s^3(2s^2+rs+r^2)\big[e_2\omega_1e_1E_{21}e_1 + E_{21}e_1e_2\omega_1e_1\big]\\
&\quad -(r+s)^2s^{-1}\big[e_1^2E_{21}e_2\omega_1 - (rs)^6E_{21}e_2\omega_1e_1^2\big]\big\}\\
&= \Delta\big\{(rs)^2(r^3-s^3)\big[(rs)^{-2}(r+s)e_1E_{21}e_2e_1 + (e_2e_1)e_1E_{21} + E_{21}e_1(r^{-3}e_1e_2)\big]\\
&\quad + s^2(2r^2+rs+s^2)\big[r^{-1}se_1e_2e_1E_{21} + e_1E_{21}(r^{-3}e_1e_2)\big]\\
&\quad - r^5s^3(2s^2+rs+r^2)\big[(e_2e_1)E_{21}e_1 + r^{-2}s^{-1}E_{21}e_1e_2e_1\big]\\
&\quad -(r+s)^2s^2\big[e_1(r^{-3}e_1e_2)E_{21} - r^5sE_{21}(e_2e_1)e_1\big]\big\}\,\omega_1\\
&= \Delta\big\{(rs)^2(r^3-s^3)\big[(rs)^{-2}(r+s)e_1E_{21}e_2e_1 + r^{-3}e_1e_2e_1E_{21} + E_{21}e_1e_2e_1\big]\\
&\quad + s^2(2r^2+rs+s^2)\big[r^{-1}se_1e_2e_1E_{21} + e_1E_{21}e_2e_1 - e_1E_{21}^2\big]\\
&\quad - r^5s^3(2s^2+rs+r^2)\big[E_{21}^2e_1 + (rs)^{-3}e_1E_{21}e_2e_1 + r^{-2}s^{-1}E_{21}e_1e_2e_1\big]\\
&\quad -(r+s)^2s^2\big[e_1e_2e_1E_{21} - e_1E_{21}^2 - r^5sE_{21}^2e_1 - r^2sE_{21}e_1e_2e_1\big]\big\}\,\omega_1\\
&= (r^3-s^3)\left(r^5s^4E_{21}^2e_1 - rs^2e_1E_{21}^2\right)\omega_1.
\end{aligned}
$$

Thus, we show $S_1=A+B+C+D=0$. This completes the proof of Lemma 3.10. □

The next identity is a consequence of Lemmas 3.9 and 3.10 and [BKL, Lemma 3.4].

LEMMA 3.12.

$$e_1E_{21}^3 - s\Delta E_{21}e_1E_{21}^2 + rs^3\Delta E_{21}^2e_1E_{21} - (rs^2)^3E_{21}^3e_1 = 0.$$

LEMMA 3.13. *$\mathcal{T}_2$ preserves the (r,s)-Serre relations $(G5)_2, (G6)_2$ into its associated object $U_{s^{-1},r^{-1}}(G_2)$.*

PROOF. For the fourth degree (r,s)-Serre relation $(G5)_2$, we have to prove that

$$\begin{aligned}(rs)^6\mathcal{T}_2(e_1)^4\mathcal{T}_2(e_2) - (rs)^3(r{+}s)(r^2{+}s^2)\mathcal{T}_2(e_1)^3\mathcal{T}_2(e_2)\mathcal{T}_2(e_1)\\ + (rs)(r^2{+}s^2)(r^2{+}rs{+}s^2)\mathcal{T}_2(e_1)^2\mathcal{T}_2(e_2)\mathcal{T}_2(e_1)^2\\ - (r{+}s)(r^2{+}s^2)\mathcal{T}_2(e_1)\mathcal{T}_2(e_2)\mathcal{T}_2(e_1)^3 + \mathcal{T}_2(e_2)\mathcal{T}_2(e_1)^4\end{aligned}$$

vanishes. By virtue of the commutative relation in (3), this is equivalent to

$$e_1\mathcal{T}_2(e_1)^3 - s\Delta\mathcal{T}_2(e_1)e_1\mathcal{T}_2(e_1)^2 + rs^3\Delta\mathcal{T}_2(e_1)^2e_1\mathcal{T}_2(e_1) - (rs^2)^3\mathcal{T}_2(e_1)^3e_1 = 0.$$

However, as $\mathcal{T}_2(e_1) = e_1e_2 - r^3e_2e_1 = (-r^3)E_{21}$, the above identity is exactly the one given by Lemma 3.12.

Similarly, we can verify that $\mathcal{T}_2$ preserves the (r,s)-Serre relation $(G6)_2$ into its associated object $U_{s^{-1},r^{-1}}(G_2)$. □

LEMMA 3.14. *$\mathcal{T}_1$ preserves the (r,s)-Serre relations $(G5)_2, (G6)_2$ into its associated object $U_{s^{-1},r^{-1}}(G_2)$.*

PROOF. For the fourth degree (r,s)-Serre relation $(G5)_2$, we have to prove that

$$\begin{aligned}(rs)^6\mathcal{T}_1(e_1)^4\mathcal{T}_1(e_2) - (rs)^3(r{+}s)(r^2{+}s^2)\mathcal{T}_1(e_1)^3\mathcal{T}_1(e_2)\mathcal{T}_1(e_1)\\ + (rs)(r^2{+}s^2)(r^2{+}rs{+}s^2)\mathcal{T}_1(e_1)^2\mathcal{T}_1(e_2)\mathcal{T}_1(e_1)^2\\ - (r{+}s)(r^2{+}s^2)\mathcal{T}_1(e_1)\mathcal{T}_1(e_2)\mathcal{T}_1(e_1)^3 + \mathcal{T}_1(e_2)\mathcal{T}_1(e_1)^4 = 0.\end{aligned}$$

In view of the commutation relation in (7), this is equivalent to

$$E_{112}\mathcal{T}_1(e_1)^3 - r\Delta\mathcal{T}_1(e_1)E_{112}\mathcal{T}_1(e_1)^2 + r^3s\Delta\mathcal{T}_1(e_1)^2E_{112}\mathcal{T}_1(e_1) - (r^2s)^3\mathcal{T}_1(e_1)^3E_{112} = 0. \tag{10}$$

We can further reduce this condition to

$$E_{12}\mathcal{T}_1(e_1)^2 - r^2(r+s)\mathcal{T}_1(e_1)E_{12}\mathcal{T}_1(e_1) + r^5s\mathcal{T}_1(e_1)^2E_{12} = 0, \tag{11}$$

as a consequence of the commutative relation

$$E_{112}\mathcal{T}_1(e_1) = rs^2\mathcal{T}_1(e_1)E_{112} + r^{-1}s(r+s)^2E_{12},$$

itself arising from the equalities $[E_{112}\, f_1] = -(r{+}s)^2E_{12}\omega_1'$ and $\omega_1'E_{112} = rs^2E_{112}\omega_1'$.

Again, since $[E_{12}, f_1] = -\Delta e_2\omega_1'$, we have

$$E_{12}\mathcal{T}_1(e_1) = r^2s\mathcal{T}_1(e_1)E_{12} + r^{-1}s\Delta e_2,$$

by which (11) is finally reduced to $e_2\mathcal{T}_1(e_1) = r^3\mathcal{T}_1(e_1)e_2$, since $\mathcal{T}_1(e_1) = -\omega_1'^{-1}f_1$.

The proof of the second part is similar. □

THEOREM 3.15. *$\mathcal{T}_1$ and $\mathcal{T}_2$ are the Lusztig's symmetries from $U_{r,s}(G_2)$ to its associated quantum group $U_{s^{-1},r^{-1}}(G_2)$ as $\mathbb{Q}$-isomorphisms, inducing the usual Lusztig's symmetries as $\mathbb{Q}(q)$-automorphisms not only on the standard quantum group $U_q(G_2)$ of Drinfel'd-Jimbo type but also on the centralizd quantum group $U_q^c(G_2)$, only when $r = q = s^{-1}$.* □

4. Appendix: Some Calculations in the proof of Proposition 2.3

For $X = f_1^4 f_2$, the relevant terms of $\Delta^{(4)}(X)$ in $(*)$ and their paring-values are as follows:

TABULAR 1

SUMMANDS	**1**
$f_1\omega_1'^3\omega_2' \otimes f_1\omega_1'^2\omega_2' \otimes f_1\omega_1'\omega_2' \otimes f_1\omega_2' \otimes f_2$	a
$\omega_1' f_1\omega_1'^2\omega_2' \otimes f_1\omega_1'^2\omega_2' \otimes f_1\omega_1'\omega_2' \otimes f_1\omega_2' \otimes f_2$	xa
$\omega_1'^2 f_1\omega_1'\omega_2' \otimes f_1\omega_1'^2\omega_2' \otimes f_1\omega_1'\omega_2' \otimes f_1\omega_2' \otimes f_2$	x^2a
$\omega_1'^3 f_1\omega_2' \otimes f_1\omega_1'^2\omega_2' \otimes f_1\omega_1'\omega_2' \otimes f_1\omega_2' \otimes f_2$	x^3a
$f_1\omega_1'^3\omega_2' \otimes \omega_1' f_1\omega_1'\omega_2' \otimes f_1\omega_1'\omega_2' \otimes f_1\omega_2' \otimes f_2$	xa
$\omega_1' f_1\omega_1'^2\omega_2' \otimes \omega_1' f_1\omega_1'\omega_2' \otimes f_1\omega_1'\omega_2' \otimes f_1\omega_2' \otimes f_2$	x^2a
$\omega_1'^2 f_1\omega_1'\omega_2' \otimes \omega_1' f_1\omega_1'\omega_2' \otimes f_1\omega_1'\omega_2' \otimes f_1\omega_2' \otimes f_2$	x^3a
$\omega_1'^3 f_1\omega_2' \otimes \omega_1' f_1\omega_1'\omega_2' \otimes f_1\omega_1'\omega_2' \otimes f_1\omega_2' \otimes f_2$	x^4a
$f_1\omega_1'^3\omega_2' \otimes \omega_1'^2 f_1\omega_2' \otimes f_1\omega_1'\omega_2' \otimes f_1\omega_2' \otimes f_2$	x^2a
$\omega_1' f_1\omega_1'^2\omega_2' \otimes \omega_1'^2 f_1\omega_2' \otimes f_1\omega_1'\omega_2' \otimes f_1\omega_2' \otimes f_2$	x^3a
$\omega_1'^2 f_1\omega_1'\omega_2' \otimes \omega_1'^2 f_1\omega_2' \otimes f_1\omega_1'\omega_2' \otimes f_1\omega_2' \otimes f_2$	x^4a
$\omega_1'^3 f_1\omega_2' \otimes \omega_1'^2 f_1\omega_2' \otimes f_1\omega_1'\omega_2' \otimes f_1\omega_2' \otimes f_2$	x^5a
$f_1\omega_1'^3\omega_2' \otimes f_1\omega_1'^2\omega_2' \otimes \omega_1' f_1\omega_2' \otimes f_1\omega_2' \otimes f_2$	xa
$\omega_1' f_1\omega_1'^2\omega_2' \otimes f_1\omega_1'^2\omega_2' \otimes \omega_1' f_1\omega_2' \otimes f_1\omega_2' \otimes f_2$	x^2a
$\omega_1'^2 f_1\omega_1'\omega_2' \otimes f_1\omega_1'^2\omega_2' \otimes \omega_1' f_1\omega_2' \otimes f_1\omega_2' \otimes f_2$	x^3a
$\omega_1'^3 f_1\omega_2' \otimes f_1\omega_1'^2\omega_2' \otimes \omega_1' f_1\omega_2' \otimes f_1\omega_2' \otimes f_2$	x^4a
$f_1\omega_1'^3\omega_2' \otimes \omega_1' f_1\omega_1'\omega_2' \otimes \omega_1' f_1\omega_2' \otimes f_1\omega_2' \otimes f_2$	x^2a
$\omega_1' f_1\omega_1'^2\omega_2' \otimes \omega_1' f_1\omega_1'\omega_2' \otimes \omega_1' f_1\omega_2' \otimes f_1\omega_2' \otimes f_2$	x^3a
$\omega_1'^2 f_1\omega_1'\omega_2' \otimes \omega_1' f_1\omega_1'\omega_2' \otimes \omega_1' f_1\omega_2' \otimes f_1\omega_2' \otimes f_2$	x^4a
$\omega_1'^3 f_1\omega_2' \otimes \omega_1' f_1\omega_1'\omega_2' \otimes \omega_1' f_1\omega_2' \otimes f_1\omega_2' \otimes f_2$	x^5a
$f_1\omega_1'^3\omega_2' \otimes \omega_1'^2 f_1\omega_2' \otimes \omega_1' f_1\omega_2' \otimes f_1\omega_2' \otimes f_2$	x^3a
$\omega_1' f_1\omega_1'^2\omega_2' \otimes \omega_1'^2 f_1\omega_2' \otimes \omega_1' f_1\omega_2' \otimes f_1\omega_2' \otimes f_2$	x^4a
$\omega_1'^2 f_1\omega_1'\omega_2' \otimes \omega_1'^2 f_1\omega_2' \otimes \omega_1' f_1\omega_2' \otimes f_1\omega_2' \otimes f_2$	x^5a
$\omega_1'^3 f_1\omega_2' \otimes \omega_1'^2 f_1\omega_2' \otimes \omega_1' f_1\omega_2' \otimes f_1\omega_2' \otimes f_2$	x^6a

SUMMANDS	**2**
$f_1\omega_1'^3\omega_2' \otimes f_1\omega_1'^2\omega_2' \otimes f_1\omega_1'\omega_2' \otimes \omega_1' f_2 \otimes f_1$	$\bar{x}a$
$\omega_1' f_1\omega_1'^2\omega_2' \otimes f_1\omega_1'^2\omega_2' \otimes f_1\omega_1'\omega_2' \otimes \omega_1' f_2 \otimes f_1$	$\bar{x}xa$
$\omega_1'^2 f_1\omega_1'\omega_2' \otimes f_1\omega_1'^2\omega_2' \otimes f_1\omega_1'\omega_2' \otimes \omega_1' f_2 \otimes f_1$	$\bar{x}x^2a$
$\omega_1'^3 f_1\omega_2' \otimes f_1\omega_1'^2\omega_2' \otimes f_1\omega_1'\omega_2' \otimes \omega_1' f_2 \otimes f_1$	$\bar{x}x^3a$
$f_1\omega_1'^3\omega_2' \otimes \omega_1' f_1\omega_1'\omega_2' \otimes f_1\omega_1'\omega_2' \otimes \omega_1' f_2 \otimes f_1$	$\bar{x}xa$
$\omega_1' f_1\omega_1'^2\omega_2' \otimes \omega_1' f_1\omega_1'\omega_2' \otimes f_1\omega_1'\omega_2' \otimes \omega_1' f_2 \otimes f_1$	$\bar{x}x^2a$
$\omega_1'^2 f_1\omega_1'\omega_2' \otimes \omega_1' f_1\omega_1'\omega_2' \otimes f_1\omega_1'\omega_2' \otimes \omega_1' f_2 \otimes f_1$	$\bar{x}x^3a$
$\omega_1'^3 f_1\omega_2' \otimes \omega_1' f_1\omega_1'\omega_2' \otimes f_1\omega_1'\omega_2' \otimes \omega_1' f_2 \otimes f_1$	$\bar{x}x^4a$
$f_1\omega_1'^3\omega_2' \otimes \omega_1'^2 f_1\omega_2' \otimes f_1\omega_1'\omega_2' \otimes \omega_1' f_2 \otimes f_1$	$\bar{x}x^2a$
$\omega_1' f_1\omega_1'^2\omega_2' \otimes \omega_1'^2 f_1\omega_2' \otimes f_1\omega_1'\omega_2' \otimes \omega_1' f_2 \otimes f_1$	$\bar{x}x^3a$
$\omega_1'^2 f_1\omega_1'\omega_2' \otimes \omega_1'^2 f_1\omega_2' \otimes f_1\omega_1'\omega_2' \otimes \omega_1' f_2 \otimes f_1$	$\bar{x}x^4a$
$\omega_1'^3 f_1\omega_2' \otimes \omega_1'^2 f_1\omega_2' \otimes f_1\omega_1'\omega_2' \otimes \omega_1' f_2 \otimes f_1$	$\bar{x}x^5a$

SUMMANDS	**2**
$f_1\omega_1'^3\omega_2' \otimes f_1\omega_1'^2\omega_2' \otimes \omega_1' f_1\omega_2' \otimes \omega_1' f_2 \otimes f_1$	$\bar{x}xa$
$\omega_1' f_1\omega_1'^2\omega_2' \otimes f_1\omega_1'^2\omega_2' \otimes \omega_1' f_1\omega_2' \otimes \omega_1' f_2 \otimes f_1$	$\bar{x}x^2a$
$\omega_1'^2 f_1\omega_1'\omega_2' \otimes f_1\omega_1'^2\omega_2' \otimes \omega_1' f_1\omega_2' \otimes \omega_1' f_2 \otimes f_1$	$\bar{x}x^3a$
$\omega_1'^3 f_1\omega_2' \otimes f_1\omega_1'^2\omega_2' \otimes \omega_1' f_1\omega_2' \otimes \omega_1' f_2 \otimes f_1$	$\bar{x}x^4a$
$f_1\omega_1'^3\omega_2' \otimes \omega_1' f_1\omega_1'\omega_2' \otimes \omega_1' f_1\omega_2' \otimes \omega_1' f_2 \otimes f_1$	$\bar{x}x^2a$
$\omega_1' f_1\omega_1'^2\omega_2' \otimes \omega_1' f_1\omega_1'\omega_2' \otimes \omega_1' f_1\omega_2' \otimes \omega_1' f_2 \otimes f_1$	$\bar{x}x^3a$
$\omega_1'^2 f_1\omega_1'\omega_2' \otimes \omega_1' f_1\omega_1'\omega_2' \otimes \omega_1' f_1\omega_2' \otimes \omega_1' f_2 \otimes f_1$	$\bar{x}x^4a$
$\omega_1'^3 f_1\omega_2' \otimes \omega_1' f_1\omega_1'\omega_2' \otimes \omega_1' f_1\omega_2' \otimes \omega_1' f_2 \otimes f_1$	$\bar{x}x^5a$
$f_1\omega_1'^3\omega_2' \otimes \omega_1'^2 f_1\omega_2' \otimes \omega_1' f_1\omega_2' \otimes \omega_1' f_2 \otimes f_1$	$\bar{x}x^3a$
$\omega_1' f_1\omega_1'^2\omega_2' \otimes \omega_1'^2 f_1\omega_2' \otimes \omega_1' f_1\omega_2' \otimes \omega_1' f_2 \otimes f_1$	$\bar{x}x^4a$
$\omega_1'^2 f_1\omega_1'\omega_2' \otimes \omega_1'^2 f_1\omega_2' \otimes \omega_1' f_1\omega_2' \otimes \omega_1' f_2 \otimes f_1$	$\bar{x}x^5a$
$\omega_1'^3 f_1\omega_2' \otimes \omega_1'^2 f_1\omega_2' \otimes \omega_1' f_1\omega_2' \otimes \omega_1' f_2 \otimes f_1$	$\bar{x}x^6a$

SUMMANDS	**3**
$f_1\omega_1'^3\omega_2' \otimes f_1\omega_1'^2\omega_2' \otimes \omega_1'^2 f_2 \otimes \omega_1' f_1 \otimes f_1$	$\bar{x}^2xa$
$\omega_1' f_1\omega_1'^2\omega_2' \otimes f_1\omega_1'^2\omega_2' \otimes \omega_1'^2 f_2 \otimes \omega_1' f_1 \otimes f_1$	$\bar{x}^2x^2a$
$\omega_1'^2 f_1\omega_1'\omega_2' \otimes f_1\omega_1'^2\omega_2' \otimes \omega_1'^2 f_2 \otimes \omega_1' f_1 \otimes f_1$	$\bar{x}^2x^3a$
$\omega_1'^3 f_1\omega_2' \otimes f_1\omega_1'^2\omega_2' \otimes \omega_1'^2 f_2 \otimes \omega_1' f_1 \otimes f_1$	$\bar{x}^2x^4a$
$f_1\omega_1'^3\omega_2' \otimes \omega_1' f_1\omega_1'\omega_2' \otimes \omega_1'^2 f_2 \otimes \omega_1' f_1 \otimes f_1$	$\bar{x}^2x^2a$
$\omega_1' f_1\omega_1'^2\omega_2' \otimes \omega_1' f_1\omega_1'\omega_2' \otimes \omega_1'^2 f_2 \otimes \omega_1' f_1 \otimes f_1$	$\bar{x}^2x^3a$
$\omega_1'^2 f_1\omega_1'\omega_2' \otimes \omega_1' f_1\omega_1'\omega_2' \otimes \omega_1'^2 f_2 \otimes \omega_1' f_1 \otimes f_1$	$\bar{x}^2x^4a$
$\omega_1'^3 f_1\omega_2' \otimes \omega_1' f_1\omega_1'\omega_2' \otimes \omega_1'^2 f_2 \otimes \omega_1' f_1 \otimes f_1$	$\bar{x}^2x^5a$
$f_1\omega_1'^3\omega_2' \otimes \omega_1'^2 f_1\omega_2' \otimes \omega_1'^2 f_2 \otimes \omega_1' f_1 \otimes f_1$	$\bar{x}^2x^3a$
$\omega_1' f_1\omega_1'^2\omega_2' \otimes \omega_1'^2 f_1\omega_2' \otimes \omega_1'^2 f_2 \otimes \omega_1' f_1 \otimes f_1$	$\bar{x}^2x^4a$
$\omega_1'^2 f_1\omega_1'\omega_2' \otimes \omega_1'^2 f_1\omega_2' \otimes \omega_1'^2 f_2 \otimes \omega_1' f_1 \otimes f_1$	$\bar{x}^2x^5a$
$\omega_1'^3 f_1\omega_2' \otimes \omega_1'^2 f_1\omega_2' \otimes \omega_1'^2 f_2 \otimes \omega_1' f_1 \otimes f_1$	$\bar{x}^2x^6a$
$f_1\omega_1'^3\omega_2' \otimes f_1\omega_1'^2\omega_2' \otimes \omega_1'^2 f_2 \otimes f_1\omega_1' \otimes f_1$	$\bar{x}^2a$
$\omega_1' f_1\omega_1'^2\omega_2' \otimes f_1\omega_1'^2\omega_2' \otimes \omega_1'^2 f_2 \otimes f_1\omega_1' \otimes f_1$	$\bar{x}^2xa$
$\omega_1'^2 f_1\omega_1'\omega_2' \otimes f_1\omega_1'^2\omega_2' \otimes \omega_1'^2 f_2 \otimes f_1\omega_1' \otimes f_1$	$\bar{x}^2x^2a$
$\omega_1'^3 f_1\omega_2' \otimes f_1\omega_1'^2\omega_2' \otimes \omega_1'^2 f_2 \otimes f_1\omega_1' \otimes f_1$	$\bar{x}^2x^3a$
$f_1\omega_1'^3\omega_2' \otimes \omega_1' f_1\omega_1'\omega_2' \otimes \omega_1'^2 f_2 \otimes f_1\omega_1' \otimes f_1$	$\bar{x}^2xa$
$\omega_1' f_1\omega_1'^2\omega_2' \otimes \omega_1' f_1\omega_1'\omega_2' \otimes \omega_1'^2 f_2 \otimes f_1\omega_1' \otimes f_1$	$\bar{x}^2x^2a$
$\omega_1'^2 f_1\omega_1'\omega_2' \otimes \omega_1' f_1\omega_1'\omega_2' \otimes \omega_1'^2 f_2 \otimes f_1\omega_1' \otimes f_1$	$\bar{x}^2x^3a$
$\omega_1'^3 f_1\omega_2' \otimes \omega_1' f_1\omega_1'\omega_2' \otimes \omega_1'^2 f_2 \otimes f_1\omega_1' \otimes f_1$	$\bar{x}^2x^4a$
$f_1\omega_1'^3\omega_2' \otimes \omega_1'^2 f_1\omega_2' \otimes \omega_1'^2 f_2 \otimes f_1\omega_1' \otimes f_1$	$\bar{x}^2x^2a$
$\omega_1' f_1\omega_1'^2\omega_2' \otimes \omega_1'^2 f_1\omega_2' \otimes \omega_1'^2 f_2 \otimes f_1\omega_1' \otimes f_1$	$\bar{x}^2x^3a$
$\omega_1'^2 f_1\omega_1'\omega_2' \otimes \omega_1'^2 f_1\omega_2' \otimes \omega_1'^2 f_2 \otimes f_1\omega_1' \otimes f_1$	$\bar{x}^2x^4a$
$\omega_1'^3 f_1\omega_2' \otimes \omega_1'^2 f_1\omega_2' \otimes \omega_1'^2 f_2 \otimes f_1\omega_1' \otimes f_1$	$\bar{x}^2x^5a$

SUMMANDS	**4**
$f_1\omega_1'^3\omega_2' \otimes \omega_1'^3 f_2 \otimes f_1\omega_1'^2 \otimes \omega_1' f_1 \otimes f_1$	$\bar{x}^3xa$
$\omega_1' f_1\omega_1'^2\omega_2' \otimes \omega_1'^3 f_2 \otimes f_1\omega_1'^2 \otimes \omega_1' f_1 \otimes f_1$	$\bar{x}^3x^2a$
$\omega_1'^2 f_1\omega_1'\omega_2' \otimes \omega_1'^3 f_2 \otimes f_1\omega_1'^2 \otimes \omega_1' f_1 \otimes f_1$	$\bar{x}^3x^3a$
$\omega_1'^3 f_1\omega_2' \otimes \omega_1'^3 f_2 \otimes f_1\omega_1'^2 \otimes \omega_1' f_1 \otimes f_1$	$\bar{x}^3x^4a$

SUMMANDS	**4**
$f_1\omega_1'^3\omega_2'\otimes\omega_1'^3f_2\otimes\omega_1'f_1\omega_1'\otimes\omega_1'f_1\otimes f_1$	$\bar{x}^3x^2a$
$\omega_1'f_1\omega_1'^2\omega_2'\otimes\omega_1'^3f_2\otimes\omega_1'f_1\omega_1'\otimes\omega_1'f_1\otimes f_1$	$\bar{x}^3x^3a$
$\omega_1'^2f_1\omega_1'\omega_2'\otimes\omega_1'^3f_2\otimes\omega_1'f_1\omega_1'\otimes\omega_1'f_1\otimes f_1$	$\bar{x}^3x^4a$
$\omega_1'^3f_1\omega_2'\otimes\omega_1'^3f_2\otimes\omega_1'f_1\omega_1'\otimes\omega_1'f_1\otimes f_1$	$\bar{x}^3x^5a$
$f_1\omega_1'^3\omega_2'\otimes\omega_1'^3f_2\otimes\omega_1'^2f_1\otimes\omega_1'f_1\otimes f_1$	$\bar{x}^3x^3a$
$\omega_1'f_1\omega_1'^2\omega_2'\otimes\omega_1'^3f_2\otimes\omega_1'^2f_1\otimes\omega_1'f_1\otimes f_1$	$\bar{x}^3x^4a$
$\omega_1'^2f_1\omega_1'\omega_2'\otimes\omega_1'^3f_2\otimes\omega_1'^2f_1\omega_1'\otimes\omega_1'f_1\otimes f_1$	$\bar{x}^3x^5a$
$\omega_1'^3f_1\omega_2'\otimes\omega_1'^3f_2\otimes\omega_1'^2f_1\otimes\omega_1'f_1\otimes f_1$	$\bar{x}^3x^6a$
$f_1\omega_1'^3\omega_2'\otimes\omega_1'^3f_2\otimes f_1\omega_1'^2\otimes f_1\omega_1'\otimes f_1$	$\bar{x}^3a$
$\omega_1'f_1\omega_1'^2\omega_2'\otimes\omega_1'^3f_2\otimes f_1\omega_1'^2\otimes f_1\omega_1'\otimes f_1$	$\bar{x}^3xa$
$\omega_1'^2f_1\omega_1'\omega_2'\otimes\omega_1'^3f_2\otimes f_1\omega_1'^2\otimes f_1\omega_1'\otimes f_1$	$\bar{x}^3x^2a$
$\omega_1'^3f_1\omega_2'\otimes\omega_1'^3f_2\otimes f_1\omega_1'^2\otimes f_1\omega_1'\otimes f_1$	$\bar{x}^3x^3a$
$f_1\omega_1'^3\omega_2'\otimes\omega_1'^3f_2\otimes\omega_1'f_1\omega_1'\otimes f_1\omega_1'\otimes f_1$	$\bar{x}^3xa$
$\omega_1'f_1\omega_1'^2\omega_2'\otimes\omega_1'^3f_2\otimes\omega_1'f_1\omega_1'\otimes f_1\omega_1'\otimes f_1$	$\bar{x}^3x^2a$
$\omega_1'^2f_1\omega_1'\omega_2'\otimes\omega_1'^3f_2\otimes\omega_1'f_1\omega_1'\otimes f_1\omega_1'\otimes f_1$	$\bar{x}^3x^3a$
$\omega_1'^3f_1\omega_2'\otimes\omega_1'^3f_2\otimes\omega_1'f_1\omega_1'\otimes f_1\omega_1'\otimes f_1$	$\bar{x}^3x^4a$
$f_1\omega_1'^3\omega_2'\otimes\omega_1'^3f_2\otimes\omega_1'^2f_1\otimes f_1\omega_1'\otimes f_1$	$\bar{x}^3x^2a$
$\omega_1'f_1\omega_1'^2\omega_2'\otimes\omega_1'^3f_2\otimes\omega_1'^2f_1\otimes f_1\omega_1'\otimes f_1$	$\bar{x}^3x^3a$
$\omega_1'^2f_1\omega_1'\omega_2'\otimes\omega_1'^3f_2\otimes\omega_1'^2f_1\omega_1'\otimes f_1\omega_1'\otimes f_1$	$\bar{x}^3x^4a$
$\omega_1'^3f_1\omega_2'\otimes\omega_1'^3f_2\otimes\omega_1'^2f_1\otimes f_1\omega_1'\otimes f_1$	$\bar{x}^3x^5a$

SUMMANDS	**5**
$\omega_1'^4f_2\otimes f_1\omega_1'^3\otimes f_1\omega_1'^2\otimes\omega_1'f_1\otimes f_1$	$\bar{x}^4xa$
$\omega_1'^4f_2\otimes\omega_1'f_1\omega_1'^2\otimes f_1\omega_1'^2\otimes\omega_1'f_1\otimes f_1$	$\bar{x}^4x^2a$
$\omega_1'^4f_2\otimes\omega_1'^2f_1\omega_1'\otimes f_1\omega_1'^2\otimes\omega_1'f_1\otimes f_1$	$\bar{x}^4x^3a$
$\omega_1'^4f_2\otimes\omega_1'^3f_1\otimes f_1\omega_1'^2\otimes\omega_1'f_1\otimes f_1$	$\bar{x}^4x^4a$
$\omega_1'^4f_2\otimes f_1\omega_1'^3\otimes\omega_1'f_1\omega_1'\otimes\omega_1'f_1\otimes f_1$	$\bar{x}^4x^2a$
$\omega_1'^4f_2\otimes\omega_1'f_1\omega_1'^2\otimes\omega_1'f_1\omega_1'\otimes\omega_1'f_1\otimes f_1$	$\bar{x}^4x^3a$
$\omega_1'^4f_2\otimes\omega_1'^2f_1\omega_1'\otimes\omega_1'f_1\omega_1'\otimes\omega_1'f_1\otimes f_1$	$\bar{x}^4x^4a$
$\omega_1'^4f_2\otimes\omega_1'^3f_1\otimes\omega_1'f_1\omega_1'\otimes\omega_1'f_1\otimes f_1$	$\bar{x}^4x^5a$
$\omega_1'^4f_2\otimes f_1\omega_1'^3\otimes\omega_1'^2f_1\otimes\omega_1'f_1\otimes f_1$	$\bar{x}^4x^3a$
$\omega_1'^4f_2\otimes\omega_1'f_1\omega_1'^2\otimes\omega_1'^2f_1\otimes\omega_1'f_1\otimes f_1$	$\bar{x}^4x^4a$
$\omega_1'^4f_2\otimes\omega_1'^2f_1\omega_1'\otimes\omega_1'^2f_1\otimes\omega_1'f_1\otimes f_1$	$\bar{x}^4x^5a$
$\omega_1'^4f_2\otimes\omega_1'^3f_1\otimes\omega_1'^2f_1\otimes\omega_1'f_1\otimes f_1$	$\bar{x}^4x^6a$
$\omega_1'^4f_2\otimes f_1\omega_1'^3\otimes f_1\omega_1'^2\otimes f_1\omega_1'\otimes f_1$	$\bar{x}^4a$
$\omega_1'^4f_2\otimes\omega_1'f_1\omega_1'^2\otimes f_1\omega_1'^2\otimes f_1\omega_1'\otimes f_1$	$\bar{x}^4xa$
$\omega_1'^4f_2\otimes\omega_1'^2f_1\omega_1'\otimes f_1\omega_1'^2\otimes f_1\omega_1'\otimes f_1$	$\bar{x}^4x^2a$
$\omega_1'^4f_2\otimes\omega_1'^3f_1\otimes f_1\omega_1'^2\otimes f_1\omega_1'\otimes f_1$	$\bar{x}^4x^3a$
$\omega_1'^4f_2\otimes f_1\omega_1'^3\otimes\omega_1'f_1\omega_1'\otimes f_1\omega_1'\otimes f_1$	$\bar{x}^4xa$
$\omega_1'^4f_2\otimes\omega_1'f_1\omega_1'^2\otimes\omega_1'f_1\omega_1'\otimes f_1\omega_1'\otimes f_1$	$\bar{x}^4x^2a$
$\omega_1'^4f_2\otimes\omega_1'^2f_1\omega_1'\otimes\omega_1'f_1\omega_1'\otimes f_1\omega_1'\otimes f_1$	$\bar{x}^4x^3a$
$\omega_1'^4f_2\otimes\omega_1'^3f_1\otimes\omega_1'f_1\omega_1'\otimes f_1\omega_1'\otimes f_1$	$\bar{x}^4x^4a$

SUMMANDS	**5**
$\omega_1'^4 f_2 \otimes f_1\omega_1'^3 \otimes \omega_1'^2 f_1 \otimes f_1\omega_1' \otimes f_1$	$\bar{x}^4x^2a$
$\omega_1'^4 f_2 \otimes \omega_1' f_1\omega_1'^2 \otimes \omega_1'^2 f_1 \otimes' f_1\omega_1 \otimes f_1$	$\bar{x}^4x^3a$
$\omega_1'^4 f_2 \otimes \omega_1'^2 f_1\omega_1' \otimes \omega_1'^2 f_1 \otimes f_1\omega_1' \otimes f_1$	$\bar{x}^4x^4a$
$\omega_1'^4 f_2 \otimes \omega_1'^3 f_1 \otimes \omega_1'^2 f_1 \otimes f_1\omega_1' \otimes f_1$	$\bar{x}^4x^5a$

The relevant terms and the pairing-values of $\Delta^{(4)}(X)$ for $X = f_2f_1^4$ in $(*)$ are as follows:

TABULAR 2

SUMMANDS	**1**
$\omega_2' f_1\omega_1'^3 \otimes \omega_2' f_1\omega_1'^2 \otimes \omega_2' f_1\omega_1' \otimes \omega_2' f_1 \otimes f_2$	ay^4
$\omega_1'\omega_2' f_1\omega_1'^2 \otimes \omega_2' f_1\omega_1'^2 \otimes \omega_2' f_1\omega_1' \otimes \omega_2' f_1 \otimes f_2$	axy^4
$\omega_1'^2\omega_2' f_1\omega_1' \otimes \omega_2' f_1\omega_1'^2 \otimes \omega_2' f_1\omega_1' \otimes \omega_2' f_1 \otimes f_2$	ax^2y^4
$\omega_1'^3\omega_2' f_1 \otimes \omega_2' f_1\omega_1'^2 \otimes \omega_2' f_1\omega_1' \otimes \omega_2' f_1 \otimes f_2$	ax^3y^4
$\omega_2' f_1\omega_1'^3 \otimes \omega_1'\omega_2' f_1\omega_1' \otimes \omega_2' f_1\omega_1' \otimes \omega_2' f_1 \otimes f_2$	axy^4
$\omega_1'\omega_2' f_1\omega_1'^2 \otimes \omega_1'\omega_2' f_1\omega_1' \otimes \omega_2' f_1\omega_1' \otimes \omega_2' f_1 \otimes f_2$	ax^2y^4
$\omega_1'^2\omega_2' f_1\omega_1' \otimes \omega_1'\omega_2' f_1\omega_1' \otimes \omega_2' f_1\omega_1' \otimes \omega_2' f_1 \otimes f_2$	ax^3y^4
$\omega_1'^3\omega_2' f_1 \otimes \omega_1'\omega_2' f_1\omega_1' \otimes \omega_2' f_1\omega_1' \otimes \omega_2' f_1 \otimes f_2$	ax^4y^4
$\omega_2' f_1\omega_1'^3 \otimes \omega_1'^2\omega_2' f_1 \otimes \omega_2' f_1\omega_1' \otimes \omega_2' f_1 \otimes f_2$	ax^2y^4
$\omega_1'\omega_2' f_1\omega_1'^2 \otimes \omega_1'^2\omega_2' f_1 \otimes \omega_2' f_1\omega_1' \otimes \omega_2' f_1 \otimes f_2$	ax^3y^4
$\omega_1'^2\omega_2' f_1\omega_1' \otimes \omega_1'^2\omega_2' f_1 \otimes \omega_2' f_1\omega_1' \otimes \omega_2' f_1 \otimes f_2$	ax^4y^4
$\omega_1'^3\omega_2' f_1 \otimes \omega_1'^2\omega_2' f_1 \otimes \omega_2' f_1\omega_1' \otimes \omega_2' f_1 \otimes f_2$	ax^5y^4
$\omega_2' f_1\omega_1'^3 \otimes \omega_2' f_1\omega_1'^2 \otimes \omega_1'\omega_2' f_1 \otimes \omega_2' f_1 \otimes f_2$	axy^4
$\omega_1'\omega_2' f_1\omega_1'^2 \otimes \omega_2' f_1\omega_1'^2 \otimes \omega_1'\omega_2' f_1 \otimes \omega_2' f_1 \otimes f_2$	ax^2y^4
$\omega_1'^2\omega_2' f_1\omega_1' \otimes \omega_2' f_1\omega_1'^2 \otimes \omega_1'\omega_2' f_1 \otimes \omega_2' f_1 \otimes f_2$	ax^3y^4
$\omega_1'^3\omega_2' f_1 \otimes \omega_2' f_1\omega_1'^2 \otimes \omega_1'\omega_2' f_1 \otimes \omega_2' f_1 \otimes f_2$	ax^4y^4
$\omega_2' f_1\omega_1'^3 \otimes \omega_1'\omega_2' f_1\omega_1' \otimes \omega_1'\omega_2' f_1 \otimes \omega_2' f_1 \otimes f_2$	ax^2y^4
$\omega_1'\omega_2' f_1\omega_1'^2 \otimes \omega_1'\omega_2' f_1\omega_1' \otimes \omega_1'\omega_2' f_1 \otimes \omega_2' f_1 \otimes f_2$	ax^3y^4
$\omega_1'^2\omega_2' f_1\omega_1' \otimes \omega_1'\omega_2' f_1\omega_1' \otimes \omega_1'\omega_2' f_1 \otimes \omega_2' f_1 \otimes f_2$	ax^4y^4
$\omega_1'^3\omega_2' f_1 \otimes \omega_1'\omega_2' f_1\omega_1' \otimes \omega_1'\omega_2' f_1 \otimes \omega_2' f_1 \otimes f_2$	ax^5y^4
$\omega_2' f_1\omega_1'^3 \otimes \omega_1'^2\omega_2' f_1 \otimes \omega_1'\omega_2' f_1 \otimes \omega_2' f_1 \otimes f_2$	ax^3y^4
$\omega_1'\omega_2' f_1\omega_1'^2 \otimes \omega_1'^2\omega_2' f_1 \otimes \omega_1'\omega_2' f_1 \otimes \omega_2' f_1 \otimes f_2$	ax^4y^4
$\omega_1'^2\omega_2' f_1\omega_1' \otimes \omega_1'^2\omega_2' f_1 \otimes \omega_1'\omega_2' f_1 \otimes \omega_2' f_1 \otimes f_2$	ax^5y^4
$\omega_1'^3\omega_2' f_1 \otimes \omega_1'^2\omega_2' f_1 \otimes \omega_1'\omega_2' f_1 \otimes \omega_2' f_1 \otimes f_2$	ax^6y^4

SUMMANDS	**2**
$\omega_2' f_1\omega_1'^3 \otimes \omega_2' f_1\omega_1'^2 \otimes \omega_2' f_1\omega_1' \otimes f_2\omega_1' \otimes f_1$	ay^3
$\omega_1'\omega_2' f_1\omega_1'^2 \otimes \omega_2' f_1\omega_1'^2 \otimes \omega_2' f_1\omega_1' \otimes f_2\omega_1' \otimes f_1$	axy^3
$\omega_1'^2\omega_2' f_1\omega_1' \otimes \omega_2' f_1\omega_1'^2 \otimes \omega_2' f_1\omega_1' \otimes f_2\omega_1' \otimes f_1$	ax^2y^3
$\omega_1'^3\omega_2' f_1 \otimes \omega_2' f_1\omega_1'^2 \otimes \omega_2' f_1\omega_1' \otimes f_2\omega_1' \otimes f_1$	ax^3y^3
$\omega_2' f_1\omega_1'^3 \otimes \omega_1'\omega_2' f_1\omega_1' \otimes \omega_2' f_1\omega_1' \otimes f_2\omega_1' \otimes f_1$	axy^3
$\omega_1'\omega_2' f_1\omega_1'^2 \otimes \omega_1'\omega_2' f_1\omega_1' \otimes \omega_2' f_1\omega_1' \otimes f_2\omega_1' \otimes f_1$	ax^2y^3
$\omega_1'^2\omega_2' f_1\omega_1' \otimes \omega_1'\omega_2' f_1\omega_1' \otimes \omega_2' f_1\omega_1' \otimes f_2\omega_1' \otimes f_1$	ax^3y^3
$\omega_1'^3\omega_2' f_1 \otimes \omega_1'\omega_2' f_1\omega_1' \otimes \omega_2' f_1\omega_1' \otimes f_2\omega_1' \otimes f_1$	ax^4y^3

SUMMANDS	**2**
$\omega_2' f_1 \omega_1'^3 \otimes \omega_1'^2 \omega_2' f_1 \otimes \omega_2' f_1 \omega_1' \otimes f_2 \omega_1' \otimes f_1$	ax^2y^3
$\omega_1' \omega_2' f_1 \omega_1'^2 \otimes \omega_1'^2 \omega_2' f_1 \otimes \omega_2' f_1 \omega_1' \otimes f_2 \omega_1' \otimes f_1$	ax^3y^3
$\omega_1'^2 \omega_2' f_1 \omega_1' \otimes \omega_1'^2 \omega_2' f_1 \otimes \omega_2' f_1 \omega_1' \otimes f_2 \omega_1' \otimes f_1$	ax^4y^3
$\omega_1'^3 \omega_2' f_1 \otimes \omega_1'^2 \omega_2' f_1 \otimes \omega_2' f_1 \omega_1' \otimes f_2 \omega_1' \otimes f_1$	ax^5y^3
$\omega_2' f_1 \omega_1'^3 \otimes \omega_2' f_1 \omega_1'^2 \otimes \omega_1' \omega_2' f_1 \otimes f_2 \omega_1' \otimes f_1$	axy^3
$\omega_1' \omega_2' f_1 \omega_1'^2 \otimes \omega_2' f_1 \omega_1'^2 \otimes \omega_1' \omega_2' f_1 \otimes f_2 \omega_1' \otimes f_1$	ax^2y^3
$\omega_1'^2 \omega_2' f_1 \omega_1' \otimes \omega_2' f_1 \omega_1'^2 \otimes \omega_1' \omega_2' f_1 \otimes f_2 \omega_1' \otimes f_1$	ax^3y^3
$\omega_1'^3 \omega_2' f_1 \otimes \omega_2' f_1 \omega_1'^2 \otimes \omega_1' \omega_2' f_1 \otimes f_2 \omega_1' \otimes f_1$	ax^4y^3
$\omega_2' f_1 \omega_1'^3 \otimes \omega_1' \omega_2' f_1 \omega_1' \otimes \omega_1' \omega_2' f_1 \otimes f_2 \omega_1' \otimes f_1$	ax^2y^3
$\omega_1' \omega_2' f_1 \omega_1'^2 \otimes \omega_1' \omega_2' f_1 \omega_1' \otimes \omega_1' \omega_2' f_1 \otimes f_2 \omega_1' \otimes f_1$	ax^3y^3
$\omega_1'^2 \omega_2' f_1 \omega_1' \otimes \omega_1' \omega_2' f_1 \omega_1' \otimes \omega_1' \omega_2' f_1 \otimes f_2 \omega_1' \otimes f_1$	ax^4y^3
$\omega_1'^3 \omega_2' f_1 \otimes \omega_1' \omega_2' f_1 \omega_1' \otimes \omega_1' \omega_2' f_1 \otimes f_2 \omega_1' \otimes f_1$	ax^5y^3
$\omega_2' f_1 \omega_1'^3 \otimes \omega_1'^2 \omega_2' f_1 \otimes \omega_1' \omega_2' f_1 \otimes f_2 \omega_1' \otimes f_1$	ax^3y^3
$\omega_1' \omega_2' f_1 \omega_1'^2 \otimes \omega_1'^2 \omega_2' f_1 \otimes \omega_1' \omega_2' f_1 \otimes f_2 \omega_1' \otimes f_1$	ax^4y^3
$\omega_1'^2 \omega_2' f_1 \omega_1' \otimes \omega_1'^2 \omega_2' f_1 \otimes \omega_1' \omega_2' f_1 \otimes f_2 \omega_1' \otimes f_1$	ax^5y^3
$\omega_1'^3 \omega_2' f_1 \otimes \omega_1'^2 \omega_2' f_1 \otimes \omega_1' \omega_2' f_1 \otimes f_2 \omega_1' \otimes f_1$	ax^6y^3

SUMMANDS	**3**
$\omega_2' f_1 \omega_1'^3 \otimes \omega_2' f_1 \omega_1'^2 \otimes f_2 \omega_1'^2 \otimes \omega_1' f_1 \otimes f_1$	axy^2
$\omega_1' \omega_2' f_1 \omega_1'^2 \otimes \omega_2' f_1 \omega_1'^2 \otimes f_2 \omega_1'^2 \otimes \omega_1' f_1 \otimes f_1$	ax^2y^2
$\omega_1'^2 \omega_2' f_1 \omega_1' \otimes \omega_2' f_1 \omega_1'^2 \otimes f_2 \omega_1'^2 \otimes \omega_1' f_1 \otimes f_1$	ax^3y^2
$\omega_1'^3 \omega_2' f_1 \otimes \omega_2' f_1 \omega_1'^2 \otimes f_2 \omega_1'^2 \otimes \omega_1' f_1 \otimes f_1$	ax^4y^2
$\omega_2' f_1 \omega_1'^3 \otimes \omega_1' \omega_2' f_1 \omega_1' \otimes f_2 \omega_1'^2 \otimes \omega_1' f_1 \otimes f_1$	ax^2y^2
$\omega_1' \omega_2' f_1 \omega_1'^2 \otimes \omega_1' \omega_2' f_1 \omega_1' \otimes f_2 \omega_1'^2 \otimes \omega_1' f_1 \otimes f_1$	ax^3y^2
$\omega_1'^2 \omega_2' f_1 \omega_1' \otimes \omega_1' \omega_2' f_1 \omega_1' \otimes f_2 \omega_1'^2 \otimes \omega_1' f_1 \otimes f_1$	ax^4y^2
$\omega_1'^3 \omega_2' f_1 \otimes \omega_1' \omega_2' f_1 \omega_1' \otimes f_2 \omega_1'^2 \otimes \omega_1' f_1 \otimes f_1$	ax^5y^2
$\omega_2' f_1 \omega_1'^3 \otimes \omega_1'^2 \omega_2' f_1 \otimes f_2 \omega_1'^2 \otimes \omega_1' f_1 \otimes f_1$	ax^3y^2
$\omega_1' \omega_2' f_1 \omega_1'^2 \otimes \omega_1'^2 \omega_2' f_1 \otimes f_2 \omega_1'^2 \otimes \omega_1' f_1 \otimes f_1$	ax^4y^2
$\omega_1'^2 \omega_2' f_1 \omega_1' \otimes \omega_1'^2 \omega_2' f_1 \otimes f_2 \omega_1'^2 \otimes \omega_1' f_1 \otimes f_1$	ax^5y^2
$\omega_1'^3 \omega_2' f_1 \otimes \omega_1'^2 \omega_2' f_1 \otimes f_2 \omega_1'^2 \otimes \omega_1' f_1 \otimes f_1$	ax^6y^2
$\omega_2' f_1 \omega_1'^3 \otimes \omega_2' f_1 \omega_1'^2 \otimes f_2 \omega_1'^2 \otimes f_1 \omega_1' \otimes f_1$	ay^2
$\omega_1' \omega_2' f_1 \omega_1'^2 \otimes \omega_2' f_1 \omega_1'^2 \otimes f_2 \omega_1'^2 \otimes f_1 \omega_1' \otimes f_1$	axy^2
$\omega_1'^2 \omega_2' f_1 \omega_1' \otimes \omega_2' f_1 \omega_1'^2 \otimes f_2 \omega_1'^2 \otimes f_1 \omega_1' \otimes f_1$	ax^2y^2
$\omega_1'^3 \omega_2' f_1 \otimes \omega_2' f_1 \omega_1'^2 \otimes f_2 \omega_1'^2 \otimes f_1 \omega_1' \otimes f_1$	ax^3y^2
$\omega_2' f_1 \omega_1'^3 \otimes \omega_1' \omega_2' f_1 \omega_1' \otimes f_2 \omega_1'^2 \otimes f_1 \omega_1' \otimes f_1$	axy^2
$\omega_1' \omega_2' f_1 \omega_1'^2 \otimes \omega_1' \omega_2' f_1 \omega_1' \otimes f_2 \omega_1'^2 \otimes f_1 \omega_1' \otimes f_1$	ax^2y^2
$\omega_1'^2 \omega_2' f_1 \omega_1' \otimes \omega_1' \omega_2' f_1 \omega_1' \otimes f_2 \omega_1'^2 \otimes f_1 \omega_1' \otimes f_1$	ax^3y^2
$\omega_1'^3 \omega_2' f_1 \otimes \omega_1' \omega_2' f_1 \omega_1' \otimes f_2 \omega_1'^2 \otimes f_1 \omega_1' \otimes f_1$	ax^4y^2
$\omega_2' f_1 \omega_1'^3 \otimes \omega_1'^2 \omega_2' f_1 \otimes f_2 \omega_1'^2 \otimes f_1 \omega_1' \otimes f_1$	ax^2y^2
$\omega_1' \omega_2' f_1 \omega_1'^2 \otimes \omega_1'^2 \omega_2' f_1 \otimes f_2 \omega_1'^2 \otimes f_1 \omega_1' \otimes f_1$	ax^3y^2
$\omega_1'^2 \omega_2' f_1 \omega_1' \otimes \omega_1'^2 \omega_2' f_1 \otimes f_2 \omega_1'^2 \otimes f_1 \omega_1' \otimes f_1$	ax^4y^2
$\omega_1'^3 \omega_2' f_1 \otimes \omega_1'^2 \omega_2' f_1 \otimes f_2 \omega_1'^2 \otimes f_1 \omega_1' \otimes f_1$	ax^5y^2

SUMMANDS	**4**
$\omega_2' f_1 \omega_1'^3 \otimes f_2 \omega_1'^3 \otimes f_1 \omega_1'^2 \otimes \omega_1' f_1 \otimes f_1$	axy
$\omega_1' \omega_2' f_1 \omega_1'^2 \otimes f_2 \omega_1'^3 \otimes f_1 \omega_1'^2 \otimes \omega_1' f_1 \otimes f_1$	ax^2y
$\omega_1'^2 \omega_2' f_1 \omega_1' \otimes f_2 \omega_1'^3 \otimes f_1 \omega_1'^2 \otimes \omega_1' f_1 \otimes f_1$	ax^3y
$\omega_1'^3 \omega_2' f_1 \otimes f_2 \omega_1'^3 \otimes f_1 \omega_1'^2 \otimes \omega_1' f_1 \otimes f_1$	ax^4y
$\omega_2' f_1 \omega_1'^3 \otimes f_2 \omega_1'^3 \otimes \omega_1' f_1 \omega_1' \otimes \omega_1' f_1 \otimes f_1$	ax^2y
$\omega_1' \omega_2' f_1 \omega_1'^2 \otimes f_2 \omega_1'^3 \otimes \omega_1' f_1 \omega_1' \otimes \omega_1' f_1 \otimes f_1$	ax^3y
$\omega_1'^2 \omega_2' f_1 \omega_1' \otimes f_2 \omega_1'^3 \otimes \omega_1' f_1 \omega_1' \otimes \omega_1' f_1 \otimes f_1$	ax^4y
$\omega_1'^3 \omega_2' f_1 \otimes f_2 \omega_1'^3 \otimes \omega_1' f_1 \omega_1' \otimes \omega_1' f_1 \otimes f_1$	ax^5y
$\omega_2' f_1 \omega_1'^3 \otimes f_2 \omega_1'^3 \otimes \omega_1'^2 f_1 \otimes \omega_1' f_1 \otimes f_1$	ax^3y
$\omega_1' \omega_2' f_1 \omega_1'^2 \otimes f_2 \omega_1'^3 \otimes \omega_1'^2 f_1 \otimes \omega_1' f_1 \otimes f_1$	ax^4y
$\omega_1'^2 \omega_2' f_1 \omega_1' \otimes f_2 \omega_1'^3 \otimes \omega_1'^2 f_1 \otimes \omega_1' f_1 \otimes f_1$	ax^5y
$\omega_1'^3 \omega_2' f_1 \otimes f_2 \omega_1'^3 \otimes \omega_1'^2 f_1 \otimes \omega_1' f_1 \otimes f_1$	ax^6y
$\omega_2' f_1 \omega_1'^3 \otimes f_2 \omega_1'^3 \otimes f_1 \omega_1'^2 \otimes f_1 \omega_1' \otimes f_1$	ay
$\omega_1' \omega_2' f_1 \omega_1'^2 \otimes f_2 \omega_1'^3 \otimes f_1 \omega_1'^2 \otimes f_1 \omega_1' \otimes f_1$	axy
$\omega_1'^2 \omega_2' f_1 \omega_1' \otimes f_2 \omega_1'^3 \otimes f_1 \omega_1'^2 \otimes f_1 \omega_1' \otimes f_1$	ax^2y
$\omega_1'^3 \omega_2' f_1 \otimes f_2 \omega_1'^3 \otimes f_1 \omega_1'^2 \otimes f_1 \omega_1' \otimes f_1$	ax^3y
$\omega_2' f_1 \omega_1'^3 \otimes f_2 \omega_1'^3 \otimes \omega_1' f_1 \omega_1' \otimes f_1 \omega_1' \otimes f_1$	axy
$\omega_1' \omega_2' f_1 \omega_1'^2 \otimes f_2 \omega_1'^3 \otimes \omega_1' f_1 \omega_1' \otimes f_1 \omega_1' \otimes f_1$	ax^2y
$\omega_1'^2 \omega_2' f_1 \omega_1' \otimes f_2 \omega_1'^3 \otimes \omega_1' f_1 \omega_1' \otimes f_1 \omega_1' \otimes f_1$	ax^3y
$\omega_1'^3 \omega_2' f_1 \otimes f_2 \omega_1'^3 \otimes \omega_1' f_1 \omega_1' \otimes f_1 \omega_1' \otimes f_1$	ax^4y
$\omega_2' f_1 \omega_1'^3 \otimes f_2 \omega_1'^3 \otimes \omega_1'^2 f_1 \otimes f_1 \omega_1' \otimes f_1$	ax^2y
$\omega_1' \omega_2' f_1 \omega_1'^2 \otimes f_2 \omega_1'^3 \otimes \omega_1'^2 f_1 \otimes f_1 \omega_1' \otimes f_1$	ax^3y
$\omega_1'^2 \omega_2' f_1 \omega_1' \otimes f_2 \omega_1'^3 \otimes \omega_1'^2 f_1 \otimes f_1 \omega_1' \otimes f_1$	ax^4y
$\omega_1'^3 \omega_2' f_1 \otimes f_2 \omega_1'^3 \otimes \omega_1'^2 f_1 \otimes f_1 \omega_1' \otimes f_1$	ax^5y

SUMMANDS	**5**
$f_2 \omega_1'^4 \otimes f_1 \omega_1'^3 \otimes f_1 \omega_1'^2 \otimes \omega_1' f_1 \otimes f_1$	ax
$f_2 \omega_1'^4 \otimes \omega_1' f_1 \omega_1'^2 \otimes f_1 \omega_1'^2 \otimes \omega_1' f_1 \otimes f_1$	ax^2
$f_2 \omega_1'^4 \otimes \omega_1'^2 f_1 \omega_1' \otimes f_1 \omega_1'^2 \otimes \omega_1' f_1 \otimes f_1$	ax^3
$f_2 \omega_1'^4 \otimes \omega_1'^3 f_1 \otimes f_1 \omega_1'^2 \otimes \omega_1' f_1 \otimes f_1$	ax^4
$f_2 \omega_1'^4 \otimes f_1 \omega_1'^3 \otimes \omega_1' f_1 \omega_1' \otimes \omega_1' f_1 \otimes f_1$	ax^2
$f_2 \omega_1'^4 \otimes \omega_1' f_1 \omega_1'^2 \otimes \omega_1' f_1 \omega_1' \otimes \omega_1' f_1 \otimes f_1$	ax^3
$f_2 \omega_1'^4 \otimes \omega_1'^2 f_1 \omega_1' \otimes \omega_1' f_1 \omega_1' \otimes \omega_1' f_1 \otimes f_1$	ax^4
$f_2 \omega_1'^4 \otimes \omega_1'^3 f_1 \otimes \omega_1' f_1 \omega_1' \otimes \omega_1' f_1 \otimes f_1$	ax^5
$f_2 \omega_1'^4 \otimes f_1 \omega_1'^3 \otimes \omega_1'^2 f_1 \otimes \omega_1' f_1 \otimes f_1$	ax^3
$f_2 \omega_1'^4 \otimes \omega_1' f_1 \omega_1'^2 \otimes \omega_1'^2 f_1 \otimes \omega_1' f_1 \otimes f_1$	ax^4
$f_2 \omega_1'^4 \otimes \omega_1'^2 f_1 \omega_1' \otimes \omega_1'^2 f_1 \otimes \omega_1' f_1 \otimes f_1$	ax^5
$f_2 \omega_1'^4 \otimes \omega_1'^3 f_1 \otimes \omega_1'^2 f_1 \otimes \omega_1' f_1 \otimes f_1$	ax^6
$f_2 \omega_1'^4 \otimes f_1 \omega_1'^3 \otimes f_1 \omega_1'^2 \otimes f_1 \omega_1' \otimes f_1$	a
$f_2 \omega_1'^4 \otimes \omega_1' f_1 \omega_1'^2 \otimes f_1 \omega_1'^2 \otimes f_1 \omega_1' \otimes f_1$	ax
$f_2 \omega_1'^4 \otimes \omega_1'^2 f_1 \omega_1' \otimes f_1 \omega_1'^2 \otimes f_1 \omega_1' \otimes f_1$	ax^2
$f_2 \omega_1'^4 \otimes \omega_1'^3 f_1 \otimes f_1 \omega_1'^2 \otimes f_1 \omega_1' \otimes f_1$	ax^3

SUMMANDS	**5**
$f_2\omega_1'^4 \otimes f_1\omega_1'^3 \otimes \omega_1' f_1\omega_1' \otimes f_1\omega_1' \otimes f_1$	ax
$f_2\omega_1'^4 \otimes \omega_1' f_1\omega_1'^2 \otimes \omega_1' f_1\omega_1' \otimes f_1\omega_1' \otimes f_1$	ax^2
$f_2\omega_1'^4 \otimes \omega_1'^2 f_1\omega_1' \otimes \omega_1' f_1\omega_1' \otimes f_1\omega_1' \otimes f_1$	ax^3
$f_2\omega_1'^4 \otimes \omega_1'^3 f_1 \otimes \omega_1' f_1\omega_1' \otimes f_1\omega_1' \otimes f_1$	ax^4
$f_2\omega_1'^4 \otimes f_1\omega_1'^3 \otimes \omega_1'^2 f_1 \otimes f_1\omega_1' \otimes f_1$	ax^2
$f_2\omega_1'^4 \otimes \omega_1' f_1\omega_1'^2 \otimes \omega_1'^2 f_1 \otimes f_1\omega_1' \otimes f_1$	ax^3
$f_2\omega_1'^4 \otimes \omega_1'^2 f_1\omega_1' \otimes \omega_1'^2 f_1 \otimes f_1\omega_1' \otimes f_1$	ax^4
$f_2\omega_1'^4 \otimes \omega_1'^3 f_1 \otimes \omega_1'^2 f_1 \otimes f_1\omega_1' \otimes f_1$	ax^5

The relevant terms and the pairing-values of $\Delta^{(4)}(X)$ for $X = f_1^2 f_2 f_1^2$ in $(*)$ are as follows:

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SUMMANDS	**1**
$f_1\omega_1'^3\omega_2' \otimes f_1\omega_1'^2\omega_2' \otimes \omega_2' f_1\omega_1' \otimes \omega_2' f_1 \otimes f_2$	ay^2
$\omega_1' f_1\omega_1'^2\omega_2' \otimes f_1\omega_1'^2\omega_2' \otimes \omega_2' f_1\omega_1' \otimes \omega_2' f_1 \otimes f_2$	axy^2
$f_1\omega_1'^3\omega_2' \otimes f_1\omega_1'^2\omega_2' \otimes \omega_1'\omega_2' f_1 \otimes \omega_2' f_1 \otimes f_2$	axy^2
$\omega_1' f_1\omega_1'^2\omega_2' \otimes f_1\omega_1'^2\omega_2' \otimes \omega_1'\omega_2' f_1 \otimes \omega_2' f_1 \otimes f_2$	ax^2y^2
$f_1\omega_1'^3\omega_2' \otimes \omega_1'\omega_2' f_1\omega_1' \otimes f_1\omega_1'\omega_2' \otimes \omega_2' f_1 \otimes f_2$	axy^2
$\omega_1' f_1\omega_1'^2\omega_2' \otimes \omega_1'\omega_2' f_1\omega_1' \otimes f_1\omega_1'\omega_2' \otimes \omega_2' f_1 \otimes f_2$	ax^2y^2
$f_1\omega_1'^3\omega_2' \otimes \omega_1'^2\omega_2' f_1 \otimes f_1\omega_1'\omega_2' \otimes \omega_2' f_1 \otimes f_2$	ax^2y^2
$\omega_1' f_1\omega_1'^2\omega_2' \otimes \omega_1'^2\omega_2' f_1 \otimes f_1\omega_1'\omega_2' \otimes \omega_2' f_1 \otimes f_2$	ax^3y^2
$\omega_1'^2\omega_2' f_1\omega_1' \otimes f_1\omega_1'^2\omega_2' \otimes f_1\omega_1'\omega_2' \otimes \omega_2' f_1 \otimes f_2$	ax^2y^2
$\omega_1'^3\omega_2' f_1 \otimes f_1\omega_1'^2\omega_2' \otimes f_1\omega_1'\omega_2' \otimes \omega_2' f_1 \otimes f_2$	ax^3y^2
$\omega_1'^2\omega_2' f_1\omega_1' \otimes \omega_1' f_1\omega_1'\omega_2' \otimes f_1\omega_1'\omega_2' \otimes \omega_2' f_1 \otimes f_2$	ax^3y^2
$\omega_1'^3\omega_2' f_1 \otimes \omega_1' f_1\omega_1'\omega_2' \otimes f_1\omega_1'\omega_2' \otimes \omega_2' f_1 \otimes f_2$	ax^4y^2
$\omega_1'^2\omega_2' f_1\omega_1' \otimes \omega_1'^2\omega_2' f_1 \otimes f_1\omega_1'\omega_2' \otimes f_1\omega_2' \otimes f_2$	ax^4y^2
$\omega_1'^3\omega_2' f_1 \otimes \omega_1'^2\omega_2' f_1 \otimes f_1\omega_1'\omega_2' \otimes f_1\omega_2' \otimes f_2$	ax^5y^2
$\omega_1'^2\omega_2' f_1\omega_1' \otimes \omega_1'^2\omega_2' f_1 \otimes \omega_1' f_1\omega_2' \otimes f_1\omega_2' \otimes f_2$	ax^5y^2
$\omega_1'^3\omega_2' f_1 \otimes \omega_1'^2\omega_2' f_1 \otimes \omega_1' f_1\omega_2' \otimes f_1\omega_2' \otimes f_2$	ax^6y^2
$f_1\omega_1'^3\omega_2' \otimes \omega_1'\omega_2' f_1\omega_1' \otimes \omega_1'\omega_2' f_1 \otimes f_1\omega_2' \otimes f_2$	ax^2y^2
$\omega_1' f_1\omega_1'^2\omega_2' \otimes \omega_1'\omega_2' f_1\omega_1' \otimes \omega_1'\omega_2' f_1 \otimes f_1\omega_2' \otimes f_2$	ax^3y^2
$f_1\omega_1'^3\omega_2' \otimes \omega_1'^2\omega_2' f_1 \otimes \omega_1'\omega_2' f_1 \otimes f_1\omega_2' \otimes f_2$	ax^3y^2
$\omega_1' f_1\omega_1'^2\omega_2' \otimes \omega_1'^2\omega_2' f_1 \otimes \omega_1'\omega_2' f_1 \otimes f_1\omega_2' \otimes f_2$	ax^4y^2
$\omega_1'^2\omega_2' f_1\omega_1' \otimes f_1\omega_1'^2\omega_2' \otimes \omega_1'\omega_2' f_1 \otimes f_1\omega_2' \otimes f_2$	ax^3y^2
$\omega_1'^3\omega_2' f_1 \otimes f_1\omega_1'^2\omega_2' \otimes \omega_1'\omega_2' f_1 \otimes f_1\omega_2' \otimes f_2$	ax^4y^2
$\omega_1'^2\omega_2' f_1\omega_1' \otimes \omega_1' f_1\omega_1'\omega_2' \otimes \omega_1'\omega_2' f_1 \otimes f_1\omega_2' \otimes f_2$	ax^4y^2
$\omega_1'^3\omega_2' f_1 \otimes \omega_1' f_1\omega_1'\omega_2' \otimes \omega_1'\omega_2' f_1 \otimes f_1\omega_2' \otimes f_2$	ax^5y^2

SUMMANDS	**2**
$f_1\omega_1'^3\omega_2' \otimes f_1\omega_1'^2\omega_2' \otimes \omega_2' f_1\omega_1' \otimes f_2\omega_1' \otimes f_1$	ay
$f_1\omega_1'^3\omega_2' \otimes f_1\omega_1'^2\omega_2' \otimes \omega_1'\omega_2' f_1 \otimes f_2\omega_1' \otimes f_1$	axy
$f_1\omega_1'^3\omega_2' \otimes \omega_1'\omega_2' f_1\omega_1' \otimes f_1\omega_1'\omega_2' \otimes f_2\omega_1' \otimes f_1$	axy

SUMMANDS	**2**
$f_1\omega'^3_1\omega'_2 \otimes \omega'^2_1\omega'_2 f_1 \otimes f_1\omega'_1\omega'_2 \otimes f_2\omega'_1 \otimes f_1$	ax^2y
$\omega'_1 f_1\omega'^2_1\omega'_2 \otimes f_1\omega'^2_1\omega'_2 \otimes \omega'_2 f_1\omega'_1 \otimes f_2\omega'_1 \otimes f_1$	axy
$\omega'_1 f_1\omega'^2_1\omega'_2 \otimes f_1\omega'^2_1\omega'_2 \otimes \omega'_1\omega'_2 f_1 \otimes f_2\omega'_1 \otimes f_1$	ax^2y
$\omega'^2_1\omega'_2 f_1\omega'_1 \otimes f_1\omega'^2_1\omega'_2 \otimes f_1\omega'_1\omega'_2 \otimes f_2\omega'_1 \otimes f_1$	ax^2y
$\omega'^3_1\omega'_2 f_1 \otimes f_1\omega'^2_1\omega'_2 \otimes f_1\omega'_1\omega'_2 \otimes f_2\omega'_1 \otimes f_1$	ax^3y
$\omega'_1 f_1\omega'^2_1\omega'_2 \otimes \omega'_1\omega'_2 f_1\omega'_1 \otimes f_1\omega'_1\omega'_2 \otimes f_2\omega'_1 \otimes f_1$	ax^2y
$\omega'_1 f_1\omega'^2_1\omega'_2 \otimes \omega'^2_1\omega'_2 f_1 \otimes f_1\omega'_1\omega'_2 \otimes f_2\omega'_1 \otimes f_1$	ax^3y
$\omega'^2_1\omega'_2 f_1\omega'_1 \otimes \omega'_1 f_1\omega'_1\omega'_2 \otimes f_1\omega'_1\omega'_2 \otimes f_2\omega'_1 \otimes f_1$	ax^3y
$\omega'^3_1\omega'_2 f_1 \otimes \omega'_1 f_1\omega'_1\omega'_2 \otimes f_1\omega'_1\omega'_2 \otimes f_2\omega'_1 \otimes f_1$	ax^4y
$f_1\omega'^3_1\omega'_2 \otimes \omega'_1\omega'_2 f_1\omega'_1 \otimes \omega'_1\omega'_2 f_1 \otimes \omega'_1 f_2 \otimes f_1$	$ax^2\bar{x}y^2$
$f_1\omega'^3_1\omega'_2 \otimes \omega'^2_1\omega'_2 f_1 \otimes \omega'_1\omega'_2 f_1 \otimes \omega'_1 f_2 \otimes f_1$	$ax^3\bar{x}y^2$
$\omega'^2_1\omega'_2 f_1\omega'_1 \otimes f_1\omega'^2_1\omega'_2 \otimes \omega'_1\omega'_2 f_1 \otimes \omega'_1 f_2 \otimes f_1$	$ax^3\bar{x}y^2$
$\omega'^3_1\omega'_2 f_1 \otimes f_1\omega'^2_1\omega'_2 \otimes \omega'_1\omega'_2 f_1 \otimes \omega'_1 f_2 \otimes f_1$	$ax^4\bar{x}y^2$
$\omega'^2_1\omega'_2 f_1\omega'_1 \otimes \omega'^2_1\omega'_2 f_1 \otimes f_1\omega'_1\omega'_2 \otimes \omega'_1 f_2 \otimes f_1$	$ax^4\bar{x}y^2$
$\omega'^3_1\omega'_2 f_1 \otimes \omega'^2_1\omega'_2 f_1 \otimes f_1\omega'_1\omega'_2 \otimes \omega'_1 f_2 \otimes f_1$	$ax^5\bar{x}y^2$
$\omega'_1 f_1\omega'^2_1\omega'_2 \otimes \omega'_1\omega'_2 f_1\omega'_1 \otimes \omega'_1\omega'_2 f_1 \otimes \omega'_1 f_2 \otimes f_1$	$ax^3\bar{x}y^2$
$\omega'_1 f_1\omega'^2_1\omega'_2 \otimes \omega'^2_1\omega'_2 f_1 \otimes \omega'_1\omega'_2 f_1 \otimes \omega'_1 f_2 \otimes f_1$	$ax^4\bar{x}y^2$
$\omega'^2_1\omega'_2 f_1\omega'_1 \otimes \omega'_1 f_1\omega'_1\omega'_2 \otimes \omega'_1\omega'_2 f_1 \otimes \omega'_1 f_2 \otimes f_1$	$ax^4\bar{x}y^2$
$\omega'^3_1\omega'_2 f_1 \otimes \omega'_1 f_1\omega'_1\omega'_2 \otimes \omega'_1\omega'_2 f_1 \otimes \omega'_1 f_2 \otimes f_1$	$ax^5\bar{x}y^2$
$\omega'^2_1\omega'_2 f_1\omega'_1 \otimes \omega'^2_1\omega'_2 f_1 \otimes \omega'_1 f_1\omega'_2 \otimes \omega'_1 f_2 \otimes f_1$	$ax^5\bar{x}y^2$
$\omega'^3_1\omega'_2 f_1 \otimes \omega'^2_1\omega'_2 f_1 \otimes \omega'_1 f_1\omega'_2 \otimes \omega'_1 f_2 \otimes f_1$	$ax^6\bar{x}y^2$

SUMMANDS	**3**
$f_1\omega'^3_1\omega'_2 \otimes f_1\omega'^2_1\omega'_2 \otimes f_2\omega'^2_1 \otimes f_1\omega'_1 \otimes f_1$	a
$f_1\omega'^3_1\omega'_2 \otimes \omega'_1\omega'_2 f_1\omega'_1 \otimes \omega'_1 f_2\omega'_1 \otimes f_1\omega'_1 \otimes f_1$	$ax\bar{x}y$
$f_1\omega'^3_1\omega'_2 \otimes \omega'^2_1\omega'_2 f_1 \otimes \omega'_1 f_2\omega'_1 \otimes f_1\omega'_1 \otimes f_1$	$ax^2\bar{x}y$
$\omega'_1 f_1\omega'^2_1\omega'_2 \otimes f_1\omega'^2_1\omega'_2 \otimes f_2\omega'^2_1 \otimes f_1\omega'_1 \otimes f_1$	ax
$\omega'^2_1\omega'_2 f_1\omega'_1 \otimes f_1\omega'^2_1\omega'_2 \otimes \omega'_1 f_2\omega'_1 \otimes f_1\omega'_1 \otimes f_1$	$ax^2\bar{x}y$
$\omega'^3_1\omega'_2 f_1 \otimes f_1\omega'^2_1\omega'_2 \otimes \omega'_1 f_2\omega'_1 \otimes f_1\omega'_1 \otimes f_1$	$ax^3\bar{x}y$
$\omega'_1 f_1\omega'^2_1\omega'_2 \otimes \omega'_1\omega'_2 f_1\omega'_1 \otimes \omega'_1 f_2\omega'_1 \otimes f_1\omega'_1 \otimes f_1$	$ax^2\bar{x}y$
$\omega'_1 f_1\omega'^2_1\omega'_2 \otimes \omega'^2_1\omega'_2 f_1 \otimes \omega'_1 f_2\omega'_1 \otimes f_1\omega'_1 \otimes f_1$	$ax^3\bar{x}y$
$\omega'^2_1\omega'_2 f_1\omega'_1 \otimes \omega'_1 f_1\omega'_1\omega'_2 \otimes \omega'_1 f_2\omega'_1 \otimes f_1\omega'_1 \otimes f_1$	$ax^3\bar{x}y$
$\omega'^3_1\omega'_2 f_1 \otimes \omega'_1 f_1\omega'_1\omega'_2 \otimes \omega'_1 f_2\omega'_1 \otimes f_1\omega'_1 \otimes f_1$	$ax^4\bar{x}y$
$\omega'^2_1\omega'_2 f_1\omega'_1 \otimes \omega'^2_1\omega'_2 f_1 \otimes \omega'^2_1 f_2 \otimes f_1\omega'_1 \otimes f_1$	$ax^4\bar{x}^2y^2$
$\omega'^3_1\omega'_2 f_1 \otimes \omega'^2_1\omega'_2 f_1 \otimes \omega'^2_1 f_2 \otimes f_1\omega'_1 \otimes f_1$	$ax^5\bar{x}^2y^2$
$f_1\omega'^3_1\omega'_2 \otimes f_1\omega'^2_1\omega'_2 \otimes f_2\omega'^2_1 \otimes \omega'_1 f_1 \otimes f_1$	ax
$f_1\omega'^3_1\omega'_2 \otimes \omega'_1\omega'_2 f_1\omega'_1 \otimes \omega'_1 f_2\omega'_1 \otimes \omega'_1 f_1 \otimes f_1$	$ax^2\bar{x}y$
$f_1\omega'^3_1\omega'_2 \otimes \omega'^2_1\omega'_2 f_1 \otimes \omega'_1 f_2\omega'_1 \otimes \omega'_1 f_1 \otimes f_1$	$ax^3\bar{x}y$
$\omega'_1 f_1\omega'^2_1\omega'_2 \otimes f_1\omega'^2_1\omega'_2 \otimes f_2\omega'^2_1 \otimes \omega'_1 f_1 \otimes f_1$	ax^2
$\omega'^2_1\omega'_2 f_1\omega'_1 \otimes f_1\omega'^2_1\omega'_2 \otimes \omega'_1 f_2\omega'_1 \otimes \omega'_1 f_1 \otimes f_1$	$ax^3\bar{x}y$
$\omega'^3_1\omega'_2 f_1 \otimes f_1\omega'^2_1\omega'_2 \otimes \omega'_1 f_2\omega'_1 \otimes \omega'_1 f_1 \otimes f_1$	$ax^4\bar{x}y$
$\omega'_1 f_1\omega'^2_1\omega'_2 \otimes \omega'_1\omega'_2 f_1\omega'_1 \otimes \omega'_1 f_2\omega'_1 \otimes \omega'_1 f_1 \otimes f_1$	$ax^3\bar{x}y$

SUMMANDS	**3**
$\omega_1'f_1\omega_1'^2\omega_2' \otimes \omega_1'^2\omega_2'f_1 \otimes \omega_1'f_2\omega_1' \otimes \omega_1'f_1 \otimes f_1$	$ax^4\bar{x}y$
$\omega_1'^2\omega_2'f_1\omega_1' \otimes \omega_1'f_1\omega_1'\omega_2' \otimes \omega_1'f_2\omega_1' \otimes \omega_1'f_1 \otimes f_1$	$ax^4\bar{x}y$
$\omega_1'^3\omega_2'f_1 \otimes \omega_1'f_1\omega_1'\omega_2' \otimes \omega_1'f_2\omega_1' \otimes \omega_1'f_1 \otimes f_1$	$ax^5\bar{x}y$
$\omega_1'^2\omega_2'f_1\omega_1' \otimes \omega_1'^2\omega_2'f_1 \otimes \omega_1'^2f_2 \otimes \omega_1'f_1 \otimes f_1$	$ax^5\bar{x}^2y^2$
$\omega_1'^3\omega_2'f_1 \otimes \omega_1'^2\omega_2'f_1 \otimes \omega_1'^2f_2 \otimes \omega_1'f_1 \otimes f_1$	$ax^6\bar{x}^2y^2$

SUMMANDS	**4**
$f_1\omega_1'^3\omega_2' \otimes \omega_1'f_2\omega_1'^2 \otimes f_1\omega_1'^2 \otimes f_1\omega_1' \otimes f_1$	$a\bar{x}$
$f_1\omega_1'^3\omega_2' \otimes \omega_1'f_2\omega_1'^2 \otimes \omega_1'f_1\omega_1' \otimes f_1\omega_1' \otimes f_1$	$ax\bar{x}$
$f_1\omega_1'^3\omega_2' \otimes \omega_1'f_2\omega_1'^2 \otimes \omega_1'^2f_1 \otimes f_1\omega_1' \otimes f_1$	$ax^2\bar{x}$
$\omega_1'f_1\omega_1'^2\omega_2' \otimes \omega_1'f_2\omega_1'^2 \otimes f_1\omega_1'^2 \otimes f_1\omega_1' \otimes f_1$	$ax\bar{x}$
$\omega_1'^2\omega_2'f_1\omega_1' \otimes \omega_1'^2f_2\omega_1' \otimes f_1\omega_1'^2 \otimes f_1\omega_1' \otimes f_1$	$ax^2\bar{x}^2y$
$\omega_1'^3\omega_2'f_1 \otimes \omega_1'^2f_2\omega_1' \otimes f_1\omega_1'^2 \otimes f_1\omega_1' \otimes f_1$	$ax^3\bar{x}^2y$
$\omega_1'f_1\omega_1'^2\omega_2' \otimes \omega_1'f_2\omega_1'^2 \otimes \omega_1'f_1\omega_1' \otimes f_1\omega_1' \otimes f_1$	$ax^2\bar{x}$
$\omega_1'f_1\omega_1'^2\omega_2' \otimes \omega_1'f_2\omega_1'^2 \otimes \omega_1'^2f_1 \otimes f_1\omega_1' \otimes f_1$	$ax^3\bar{x}$
$\omega_1'^2\omega_2'f_1\omega_1' \otimes \omega_1'^2f_2\omega_1' \otimes \omega_1'f_1\omega_1' \otimes f_1\omega_1' \otimes f_1$	$ax^3\bar{x}^2y$
$\omega_1'^3\omega_2'f_1 \otimes \omega_1'^2f_2\omega_1' \otimes \omega_1'f_1\omega_1' \otimes f_1\omega_1' \otimes f_1$	$ax^4\bar{x}^2y$
$\omega_1'^2\omega_2'f_1\omega_1' \otimes \omega_1'^2f_2\omega_1' \otimes \omega_1'^2f_1 \otimes f_1\omega_1' \otimes f_1$	$ax^4\bar{x}^2y$
$\omega_1'^3\omega_2'f_1 \otimes \omega_1'^2f_2\omega_1' \otimes \omega_1'^2f_1 \otimes f_1\omega_1' \otimes f_1$	$ax^4\bar{x}^2y$
$f_1\omega_1'^3\omega_2' \otimes \omega_1'f_2\omega_1'^2 \otimes f_1\omega_1'^2 \otimes \omega_1'f_1 \otimes f_1$	$ax\bar{x}$
$f_1\omega_1'^3\omega_2' \otimes \omega_1'f_2\omega_1'^2 \otimes \omega_1'f_1\omega_1' \otimes \omega_1'f_1 \otimes f_1$	$ax^2\bar{x}$
$f_1\omega_1'^3\omega_2' \otimes \omega_1'f_2\omega_1'^2 \otimes \omega_1'^2f_1 \otimes \omega_1'f_1 \otimes f_1$	$ax^3\bar{x}$
$\omega_1'f_1\omega_1'^2\omega_2' \otimes \omega_1'f_2\omega_1'^2 \otimes f_2\omega_1'^2 \otimes \omega_1'f_1 \otimes f_1$	$ax^2\bar{x}$
$\omega_1'^2\omega_2'f_1\omega_1' \otimes \omega_1'^2f_2\omega_1' \otimes f_1\omega_1'^2 \otimes \omega_1'f_1 \otimes f_1$	$ax^3\bar{x}^2y$
$\omega_1'^3\omega_2'f_1 \otimes \omega_1'^2f_2\omega_1' \otimes f_1\omega_1'^2 \otimes \omega_1'f_1 \otimes f_1$	$ax^4\bar{x}^2y$
$\omega_1'f_1\omega_1'^2\omega_2' \otimes \omega_1'f_2\omega_1'^2 \otimes \omega_1'f_1\omega_1' \otimes \omega_1'f_1 \otimes f_1$	$ax^3\bar{x}$
$\omega_1'f_1\omega_1'^2\omega_2' \otimes \omega_1'f_2\omega_1'^2 \otimes \omega_1'^2f_1 \otimes \omega_1'f_1 \otimes f_1$	$ax^4\bar{x}$
$\omega_1'^2\omega_2'f_1\omega_1' \otimes \omega_1'^2f_2\omega_1' \otimes \omega_1'f_1\omega_1' \otimes \omega_1'f_1 \otimes f_1$	$ax^4\bar{x}^2y$
$\omega_1'^3\omega_2'f_1 \otimes \omega_1'^2f_2\omega_1' \otimes \omega_1'f_1\omega_1' \otimes \omega_1'f_1 \otimes f_1$	$ax^5\bar{x}^2y$
$\omega_1'^2\omega_2'f_1\omega_1' \otimes \omega_1'^2f_2\omega_1' \otimes \omega_1'^2f_1 \otimes \omega_1'f_1 \otimes f_1$	$ax^5\bar{x}^2y$
$\omega_1'^3\omega_2'f_1 \otimes \omega_1'^2f_2\omega_1' \otimes \omega_1'^2f_1 \otimes \omega_1'f_1 \otimes f_1$	$ax^6\bar{x}^2y$

SUMMANDS	**5**
$\omega_1'^2f_2\omega_1'^2 \otimes f_1\omega_1'^3 \otimes f_1\omega_1'^2 \otimes f_1\omega_1' \otimes f_1$	$a\bar{x}^2$
$\omega_1'^2f_2\omega_1'^2 \otimes f_1\omega_1'^3 \otimes \omega_1'f_1\omega_1' \otimes f_1\omega_1' \otimes f_1$	$ax\bar{x}^2$
$\omega_1'^2f_2\omega_1'^2 \otimes f_1\omega_1'^3 \otimes \omega_1'^2f_1\omega_1' \otimes f_1\omega_1' \otimes f_1$	$ax^2\bar{x}^2$
$\omega_1'^2f_2\omega_1'^2 \otimes \omega_1'f_1\omega_1'^2 \otimes f_1\omega_1'^2 \otimes f_1\omega_1' \otimes f_1$	$ax\bar{x}^2$
$\omega_1'^2f_2\omega_1'^2 \otimes \omega_1'^2f_1\omega_1' \otimes f_1\omega_1'^2 \otimes f_1\omega_1' \otimes f_1$	$ax^2\bar{x}^2$
$\omega_1'^2f_2\omega_1'^2 \otimes \omega_1'^3f_1 \otimes f_1\omega_1'^2 \otimes f_1\omega_1' \otimes f_1$	$ax^3\bar{x}^2$
$\omega_1'^2f_2\omega_1'^2\omega_2' \otimes \omega_1'f_1\omega_1'^2 \otimes \omega_1'f_1\omega_1' \otimes f_1\omega_1' \otimes f_1$	$ax^2\bar{x}^2$
$\omega_1'^2f_2\omega_1'^2\omega_2' \otimes \omega_1'f_1\omega_1'^2 \otimes \omega_1'^2f_1 \otimes f_1\omega_1' \otimes f_1$	$ax^3\bar{x}^2$
$\omega_1'^2f_2\omega_1'^2 \otimes \omega_1'^2f_1\omega_1' \otimes \omega_1'f_1\omega_1' \otimes f_1\omega_1' \otimes f_1$	$ax^3\bar{x}^2$
$\omega_1'^2f_2\omega_1'^2 \otimes \omega_1'^3f_1 \otimes \omega_1'f_1\omega_1' \otimes f_1\omega_1' \otimes f_1$	$ax^4\bar{x}^2$

SUMMANDS	**5**
$\omega_1'^2 f_2\omega_1'^2 \otimes \omega_1'^2 f_1\omega_1' \otimes \omega_1'^2 f_1 \otimes f_1\omega_1' \otimes f_1$	$ax^4\bar{x}^2$
$\omega_1'^2 f_2\omega_1'^2 \otimes \omega_1'^3 f_1 \otimes \omega_1'^2 f_1 \otimes f_1\omega_1' \otimes f_1$	$ax^5\bar{x}^2$
$\omega_1'^2 f_2\omega_1'^2 \otimes f_1\omega_1'^3 \otimes f_1\omega_1'^2 \otimes \omega_1' f_1 \otimes f_1$	$ax\bar{x}^2$
$\omega_1'^2 f_2\omega_1'^2 \otimes f_1\omega_1'^3 \otimes \omega_1' f_1\omega_1' \otimes \omega_1' f_1 \otimes f_1$	$ax^2\bar{x}^2$
$\omega_1'^2 f_2\omega_1'^2 \otimes f_1\omega_1'^3 \otimes \omega_1'^2 f_1 \otimes \omega_1' f_1 \otimes f_1$	$ax^3\bar{x}^2$
$\omega_1'^2 f_2\omega_1'^2 \otimes \omega_1' f_1\omega_1'^2 \otimes f_1\omega_1'^2 \otimes \omega_1' f_1 \otimes f_1$	$ax^2\bar{x}^2$
$\omega_1'^2 f_2\omega_1'^2 \otimes \omega_1'^2 f_1\omega_1' \otimes f_1\omega_1'^2 \otimes \omega_1' f_1 \otimes f_1$	$ax^3\bar{x}^2$
$\omega_1'^2 f_2\omega_1'^2 \otimes \omega_1'^3 f_1 \otimes f_1\omega_1'^2 \otimes \omega_1' f_1 \otimes f_1$	$ax^4\bar{x}^2$
$\omega_1'^2 f_2\omega_1'^2 \otimes \omega_1' f_1\omega_1'^2 \otimes \omega_1' f_1\omega_1' \otimes \omega_1' f_1 \otimes f_1$	$ax^3\bar{x}^2$
$\omega_1'^2 f_2\omega_1'^2 \otimes \omega_1' f_1\omega_1'^2 \otimes \omega_1'^2 f_1 \otimes \omega_1' f_1 \otimes f_1$	$ax^4\bar{x}^2$
$\omega_1'^2 f_2\omega_1'^2 \otimes \omega_1'^2 f_1\omega_1' \otimes \omega_1' f_1\omega_1' \otimes \omega_1' f_1 \otimes f_1$	$ax^4\bar{x}^2$
$\omega_1'^2 f_2\omega_1'^2 \otimes \omega_1'^3 f_1 \otimes \omega_1' f_1\omega_1' \otimes \omega_1' f_1 \otimes f_1$	$ax^5\bar{x}^2$
$\omega_1'^2 f_2\omega_1'^2 \otimes \omega_1'^2 f_1\omega_1' \otimes \omega_1'^2 f_1 \otimes \omega_1' f_1 \otimes f_1$	$ax^5\bar{x}^2$
$\omega_1'^2 f_2\omega_1'^2 \otimes \omega_1'^3 f_1 \otimes \omega_1'^2 f_1 \otimes \omega_1' f_1 \otimes f_1$	$ax^6\bar{x}^2$

The relevant terms and the pairing-values of $\Delta^{(4)}(X)$ for $X = f_1^3 f_2 f_1$ in $(*)$ are as follows:

TABULAR 4

SUMMANDS	**1**
$f_1\omega_1'^3\omega_2' \otimes f_1\omega_1'^2\omega_2' \otimes f_1\omega_2'\omega_1' \otimes \omega_2' f_1 \otimes f_2$	ay
$f_1\omega_1'^3\omega_2' \otimes \omega_1' f_1\omega_1'\omega_2' \otimes f_1\omega_2'\omega_1' \otimes \omega_2' f_1 \otimes f_2$	axy
$\omega_1' f_1\omega_1'^2\omega_2' \otimes f_1\omega_1'^2\omega_2' \otimes f_1\omega_2'\omega_1' \otimes \omega_2' f_1 \otimes f_2$	axy
$\omega_1'^2 f_1\omega_1'\omega_2' \otimes f_1\omega_1'^2\omega_2' \otimes f_1\omega_2'\omega_1' \otimes \omega_2' f_1 \otimes f_2$	ax^2y
$\omega_1' f_1\omega_1'^2\omega_2' \otimes \omega_1' f_1\omega_1'\omega_2' \otimes f_1\omega_2'\omega_1' \otimes \omega_2' f_1 \otimes f_2$	ax^2y
$\omega_1'^2 f_1\omega_1'\omega_2' \otimes \omega_1' f_1\omega_1'\omega_2' \otimes f_1\omega_2'\omega_1' \otimes \omega_2' f_1 \otimes f_2$	ax^3y
$f_1\omega_1'^3\omega_2' \otimes f_1\omega_1'^2\omega_2' \otimes \omega_1'\omega_2' f_1 \otimes f_1\omega_2' \otimes f_2$	axy
$f_1\omega_1'^3\omega_2' \otimes \omega_1' f_1\omega_1'\omega_2' \otimes \omega_1'\omega_2' f_1 \otimes f_1\omega_2' \otimes f_2$	ax^2y
$\omega_1' f_1\omega_1'^2\omega_2' \otimes f_1\omega_1'^2\omega_2' \otimes \omega_1'\omega_2' f_1 \otimes f_1\omega_2' \otimes f_2$	ax^2y
$\omega_1'^2 f_1\omega_1'\omega_2' \otimes f_1\omega_1'^2\omega_2' \otimes \omega_1'\omega_2' f_1 \otimes f_1\omega_2' \otimes f_2$	ax^3y
$\omega_1' f_1\omega_1'^2\omega_2' \otimes \omega_1' f_1\omega_1'\omega_2' \otimes \omega_1'\omega_2' f_1 \otimes f_1\omega_2' \otimes f_2$	ax^3y
$\omega_1'^2 f_1\omega_1'\omega_2' \otimes \omega_1' f_1\omega_1'\omega_2' \otimes \omega_1'\omega_2' f_1 \otimes f_1\omega_2' \otimes f_2$	ax^4y
$f_1\omega_1'^3\omega_2' \otimes \omega_1'^2\omega_2' f_1 \otimes f_1\omega_1'\omega_2' \otimes f_1\omega_2' \otimes f_2$	ax^2y
$\omega_1'^3\omega_2' f_1 \otimes f_1\omega_1'^2\omega_2' \otimes f_1\omega_1'\omega_2' \otimes f_1\omega_2' \otimes f_2$	ax^3y
$\omega_1' f_1\omega_1'^2\omega_2' \otimes \omega_1'^2\omega_2' f_1 \otimes f_1\omega_1'\omega_2' \otimes f_1\omega_2' \otimes f_2$	ax^3y
$\omega_1'^3\omega_2' f_1 \otimes \omega_1' f_1\omega_1'\omega_2' \otimes f_1\omega_1'\omega_2' \otimes f_1\omega_2' \otimes f_2$	ax^4y
$\omega_1'^2 f_1\omega_1'\omega_2' \otimes \omega_1'^2\omega_2' f_1 \otimes f_1\omega_1'\omega_2' \otimes f_1\omega_2' \otimes f_2$	ax^4y
$\omega_1'^3\omega_2' f_1 \otimes \omega_1'^2 f_1\omega_2' \otimes f_1\omega_1'\omega_2' \otimes f_1\omega_2' \otimes f_2$	ax^5y
$f_1\omega_1'^3\omega_2' \otimes \omega_1'^2\omega_2' f_1 \otimes \omega_1' f_1\omega_2' \otimes f_1\omega_2' \otimes f_2$	ax^3y
$\omega_1'^3\omega_2' f_1 \otimes f_1\omega_1'^2\omega_2' \otimes \omega_1' f_1\omega_2' \otimes f_1\omega_2' \otimes f_2$	ax^4y
$\omega_1' f_1\omega_1'^2\omega_2' \otimes \omega_1'^2\omega_2' f_1 \otimes \omega_1' f_1\omega_2' \otimes f_1\omega_2' \otimes f_2$	ax^4y
$\omega_1'^3\omega_2' f_1 \otimes \omega_1' f_1\omega_1'\omega_2' \otimes \omega_1' f_1\omega_2' \otimes f_1\omega_2' \otimes f_2$	ax^5y

SUMMANDS	**1**
$\omega_1'^2 f_1\omega_1'\omega_2' \otimes \omega_1'^2\omega_2' f_1 \otimes \omega_1' f_1\omega_2' \otimes f_1\omega_2' \otimes f_2$	ax^5y
$\omega_1'^3\omega_2' f_1 \otimes \omega_1'^2 f_1\omega_2' \otimes \omega_1' f_1\omega_2' \otimes f_1\omega_2' \otimes f_2$	ax^6y

SUMMANDS	**2**
$f_1\omega_1'^3\omega_2' \otimes f_1\omega_1'^2\omega_2' \otimes f_1\omega_1'\omega_2' \otimes f_2\omega_1' \otimes f_1$	a
$f_1\omega_1'^3\omega_2' \otimes \omega_1' f_1\omega_1'\omega_2' \otimes f_1\omega_1'\omega_2' \otimes f_2\omega_1' \otimes f_1$	ax
$\omega_1' f_1\omega_1'^2\omega_2' \otimes f_1\omega_1'^2\omega_2' \otimes f_1\omega_1'\omega_2' \otimes f_2\omega_1' \otimes f_1$	ax
$\omega_1'^2 f_1\omega_1'\omega_2' \otimes f_1\omega_1'^2\omega_2' \otimes f_1\omega_1'\omega_2' \otimes f_2\omega_1' \otimes f_1$	ax^2
$\omega_1' f_1\omega_1'^2\omega_2' \otimes \omega_1' f_1\omega_1'\omega_2' \otimes f_1\omega_1'\omega_2' \otimes f_2\omega_1' \otimes f_1$	ax^2
$\omega_1'^2 f_1\omega_1'\omega_2' \otimes \omega_1' f_1\omega_1'\omega_2' \otimes f_1\omega_1'\omega_2' \otimes f_2\omega_1' \otimes f_1$	ax^3
$f_1\omega_1'^3\omega_2' \otimes f_1\omega_1'^2\omega_2' \otimes \omega_1'\omega_2' f_1 \otimes \omega_1' f_2 \otimes f_1$	$ax\bar{x}y$
$f_1\omega_1'^3\omega_2' \otimes \omega_1'^2\omega_2' f_1 \otimes f_1\omega_1'\omega_2' \otimes \omega_1' f_2 \otimes f_1$	$ax^2\bar{x}y$
$f_1\omega_1'^3\omega_2' \otimes \omega_1' f_1\omega_1'\omega_2' \otimes \omega_1'\omega_2' f_1 \otimes \omega_1' f_2 \otimes f_1$	$ax^2\bar{x}y$
$f_1\omega_1'^3\omega_2' \otimes \omega_1'^2\omega_2' f_1 \otimes \omega_1' f_1\omega_2' \otimes \omega_1' f_2 \otimes f_1$	$ax^3\bar{x}y$
$\omega_1' f_1\omega_1'^2\omega_2' \otimes f_1\omega_1'^2\omega_2' \otimes \omega_1'\omega_2' f_1 \otimes \omega_1' f_2 \otimes f_1$	$ax^2\bar{x}y$
$\omega_1'^3\omega_2' f_1 \otimes f_1\omega_1'^2\omega_2' \otimes f_1\omega_1'\omega_2' \otimes \omega_1' f_2 \otimes f_1$	$ax^3\bar{x}y$
$\omega_1'^2 f_1\omega_1'\omega_2' \otimes f_1\omega_1'^2\omega_2' \otimes \omega_1'\omega_2' f_1 \otimes \omega_1' f_2 \otimes f_1$	$ax^3\bar{x}y$
$\omega_1'^3\omega_2' f_1 \otimes f_1\omega_1'^2\omega_2' \otimes \omega_1' f_1\omega_2' \otimes \omega_1' f_2 \otimes f_1$	$ax^4\bar{x}y$
$\omega_1' f_1\omega_1'^2\omega_2' \otimes \omega_1'^2\omega_2' f_1 \otimes f_1\omega_1'\omega_2' \otimes \omega_1' f_2 \otimes f_1$	$ax^3\bar{x}y$
$\omega_1'^3\omega_2' f_1 \otimes \omega_1' f_1\omega_1'\omega_2' \otimes f_1\omega_1'\omega_2' \otimes \omega_1' f_2 \otimes f_1$	$ax^4\bar{x}y$
$\omega_1'^2 f_1\omega_1'\omega_2' \otimes \omega_1'^2\omega_2' f_1 \otimes f_1\omega_1'\omega_2' \otimes \omega_1' f_2 \otimes f_1$	$ax^4\bar{x}y$
$\omega_1'^3\omega_2' f_1 \otimes \omega_1'^2 f_1\omega_2' \otimes f_1\omega_1'\omega_2' \otimes \omega_1' f_2 \otimes f_1$	$ax^5\bar{x}y$
$\omega_1' f_1\omega_1'^2\omega_2' \otimes \omega_1' f_1\omega_1'\omega_2' \otimes \omega_1'\omega_2' f_1 \otimes \omega_1' f_2 \otimes f_1$	$ax^3\bar{x}y$
$\omega_1' f_1\omega_1'^2\omega_2' \otimes \omega_1'^2 f_1\omega_2' \otimes \omega_1' f_1\omega_2' \otimes \omega_1' f_2 \otimes f_1$	$ax^4\bar{x}y$
$\omega_1'^2 f_1\omega_1'\omega_2' \otimes \omega_1' f_1\omega_1'\omega_2' \otimes \omega_1'\omega_2' f_1 \otimes \omega_1' f_2 \otimes f_1$	$ax^4\bar{x}y$
$\omega_1'^3\omega_2' f_1 \otimes \omega_1' f_1\omega_1'\omega_2' \otimes \omega_1' f_1\omega_2' \otimes \omega_1' f_2 \otimes f_1$	$ax^5\bar{x}y$
$\omega_1'^2 f_1\omega_1'\omega_2' \otimes \omega_1'^2\omega_2' f_1 \otimes \omega_1' f_1\omega_2' \otimes \omega_1' f_2 \otimes f_1$	$ax^5\bar{x}y$
$\omega_1'^3\omega_2' f_1 \otimes \omega_1'^2 f_1\omega_2' \otimes \omega_1' f_1\omega_2' \otimes \omega_1' f_2 \otimes f_1$	$ax^6\bar{x}y$

SUMMANDS	**3**
$f_1\omega_1'^3\omega_2' \otimes f_1\omega_1'^2\omega_2' \otimes \omega_1' f_2\omega_1' \otimes f_1\omega_1' \otimes f_1$	$a\bar{x}$
$f_1\omega_1'^3\omega_2' \otimes \omega_1' f_1\omega_1'\omega_2' \otimes \omega_1' f_2\omega_1' \otimes f_1\omega_1' \otimes f_1$	$ax\bar{x}$
$f_1\omega_1'^3\omega_2' \otimes \omega_1'^2\omega_2' f_1 \otimes \omega_1'^2 f_2 \otimes f_1\omega_1' \otimes f_1$	$ax^2\bar{x}^2y$
$\omega_1' f_1\omega_1'^2\omega_2' \otimes f_1\omega_1'^2\omega_2' \otimes \omega_1' f_2\omega_1' \otimes f_1\omega_1' \otimes f_1$	$ax\bar{x}$
$\omega_1'^2 f_1\omega_1'\omega_2' \otimes f_1\omega_1'^2\omega_2' \otimes \omega_1' f_2\omega_1' \otimes f_1\omega_1' \otimes f_1$	$ax^2\bar{x}$
$\omega_1'^3\omega_2' f_1 \otimes f_1\omega_1'^2\omega_2' \otimes \omega_1'^2 f_2 \otimes f_1\omega_1' \otimes f_1$	$ax^3\bar{x}^2y$
$\omega_1' f_1\omega_1'^2\omega_2' \otimes \omega_1' f_1\omega_1'\omega_2' \otimes \omega_1' f_2\omega_1' \otimes f_1\omega_1' \otimes f_1$	$ax^2\bar{x}$
$\omega_1' f_1\omega_1'^2\omega_2' \otimes \omega_1'^2\omega_2' f_1 \otimes \omega_1'^2 f_2 \otimes f_1\omega_1' \otimes f_1$	$ax^3\bar{x}^2y$
$\omega_1'^2 f_1\omega_1'\omega_2' \otimes \omega_1' f_1\omega_1'\omega_2' \otimes \omega_1' f_2\omega_1' \otimes f_1\omega_1' \otimes f_1$	$ax^3\bar{x}$
$\omega_1'^3\omega_2' f_1 \otimes \omega_1' f_1\omega_1'\omega_2' \otimes \omega_1'^2 f_2 \otimes f_1\omega_1' \otimes f_1$	$ax^4\bar{x}^2y$
$\omega_1'^2 f_1\omega_1'\omega_2' \otimes \omega_1'^2\omega_2' f_1 \otimes \omega_1'^2 f_2 \otimes f_1\omega_1' \otimes f_1$	$ax^4\bar{x}^2y$
$\omega_1'^3\omega_2' f_1 \otimes \omega_1'^2 f_1\omega_2' \otimes \omega_1'^2 f_2 \otimes f_1\omega_1' \otimes f_1$	$ax^5\bar{x}^2y$

SUMMANDS	**3**
$f_1\omega_1'^3\omega_2' \otimes f_1\omega_1'^2\omega_2' \otimes \omega_1' f_2\omega_1' \otimes \omega_1' f_1 \otimes f_1$	$ax\bar{x}$
$f_1\omega_1'^3\omega_2' \otimes \omega_1' f_1\omega_1'\omega_2' \otimes \omega_1' f_2\omega_1' \otimes \omega_1' f_1 \otimes f_1$	$ax^2\bar{x}$
$f_1\omega_1'^3\omega_2' \otimes \omega_1'^2\omega_2' f_1 \otimes \omega_1'^2 f_2 \otimes \omega_1' f_1 \otimes f_1$	$ax^3\bar{x}^2y$
$\omega_1' f_1\omega_1'^2\omega_2' \otimes f_1\omega_1'^2\omega_2' \otimes \omega_1' f_2\omega_1' \otimes \omega_1' f_1 \otimes f_1$	$ax^2\bar{x}$
$\omega_1'^2 f_1\omega_1'\omega_2' \otimes f_1\omega_1'^2\omega_2' \otimes \omega_1' f_2\omega_1' \otimes \omega_1' f_1 \otimes f_1$	$ax^3\bar{x}$
$\omega_1'^3\omega_2' f_1 \otimes f_1\omega_1'^2\omega_2' \otimes \omega_1'^2 f_2 \otimes \omega_1' f_1 \otimes f_1$	$ax^4\bar{x}^2y$
$\omega_1' f_1\omega_1'^2\omega_2' \otimes \omega_1' f_1\omega_1'\omega_2' \otimes \omega_1' f_2\omega_1' \otimes \omega_1' f_1 \otimes f_1$	$ax^3\bar{x}$
$\omega_1' f_1\omega_1'^2\omega_2' \otimes \omega_1'^2\omega_2' f_1 \otimes \omega_1'^2 f_2 \otimes \omega_1' f_1 \otimes f_1$	$ax^4\bar{x}^2y$
$\omega_1'^2 f_1\omega_1'\omega_2' \otimes \omega_1' f_1\omega_1'\omega_2' \otimes \omega_1' f_2\omega_1' \otimes \omega_1' f_1 \otimes f_1$	$ax^4\bar{x}$
$\omega_1'^3\omega_2' f_1 \otimes \omega_1' f_1\omega_1'\omega_2' \otimes \omega_1'^2 f_2 \otimes \omega_1' f_1 \otimes f_1$	$ax^5\bar{x}^2y$
$\omega_1'^2 f_1\omega_1'\omega_2' \otimes \omega_1'^2\omega_2' f_1 \otimes \omega_1'^2 f_2 \otimes \omega_1' f_1 \otimes f_1$	$ax^5\bar{x}^2y$
$\omega_1'^3\omega_2' f_1 \otimes \omega_1'^2\omega_2' f_1 \otimes \omega_1'^2 f_2 \otimes \omega_1' f_1 \otimes f_1$	$ax^6\bar{x}^2y$

SUMMANDS	**4**
$f_1\omega_1'^3\omega_2' \otimes \omega_1'^2 f_2\omega_1' \otimes f_1\omega_1'^2 \otimes f_1\omega_1' \otimes f_1$	$a\bar{x}^2$
$f_1\omega_1'^3\omega_2' \otimes \omega_1'^2 f_2\omega_1' \otimes \omega_1' f_1\omega_1' \otimes f_1\omega_1' \otimes f_1$	$ax\bar{x}^2$
$f_1\omega_1'^3\omega_2' \otimes \omega_1'^2 f_2\omega_1' \otimes \omega_1'^2 f_1 \otimes f_1\omega_1' \otimes f_1$	$ax^2\bar{x}^2$
$\omega_1' f_1\omega_1'^2\omega_2' \otimes \omega_1'^2 f_2\omega_1' \otimes f_1\omega_1'^2 \otimes f_1\omega_1' \otimes f_1$	$ax\bar{x}^2$
$\omega_1'^2 f_1\omega_1'\omega_2' \otimes \omega_1'^2 f_2\omega_1' \otimes f_1\omega_1'^2 \otimes f_1\omega_1' \otimes f_1$	$ax^2\bar{x}^2$
$\omega_1'^3\omega_2' f_1 \otimes \omega_1'^3 f_2 \otimes f_1\omega_1'^2 \otimes f_1\omega_1' \otimes f_1$	$ax^3\bar{x}^3y$
$\omega_1' f_1\omega_1'^2\omega_2' \otimes \omega_1'^2 f_2\omega_1' \otimes \omega_1' f_1\omega_1' \otimes f_1\omega_1' \otimes f_1$	$ax^2\bar{x}^2$
$\omega_1' f_1\omega_1'^2\omega_2' \otimes \omega_1'^2 f_2\omega_1' \otimes \omega_1'^2 f_1 \otimes f_1\omega_1' \otimes f_1$	$ax^3\bar{x}^2$
$\omega_1'^2 f_1\omega_1'\omega_2' \otimes \omega_1'^2 f_2\omega_1' \otimes \omega_1' f_1\omega_1' \otimes f_1\omega_1' \otimes f_1$	$ax^3\bar{x}^2$
$\omega_1'^3\omega_2' f_1 \otimes \omega_1'^3 f_2 \otimes \omega_1' f_1\omega_1' \otimes f_1\omega_1' \otimes f_1$	$ax^4\bar{x}^3y$
$\omega_1'^2 f_1\omega_1'\omega_2' \otimes \omega_1'^2 f_2\omega_1' \otimes \omega_1'^2 f_1 \otimes f_1\omega_1' \otimes f_1$	$ax^4\bar{x}^2$
$f_1\omega_1'^3\omega_2' \otimes \omega_1'^2 f_2\omega_1' \otimes f_1\omega_1'^2 \otimes \omega_1' f_1 \otimes f_1$	$ax\bar{x}^2$
$f_1\omega_1'^3\omega_2' \otimes \omega_1'^2 f_2\omega_1' \otimes \omega_1' f_1\omega_1' \otimes \omega_1' f_1 \otimes f_1$	$ax^2\bar{x}^2$
$f_1\omega_1'^3\omega_2' \otimes \omega_1'^2 f_2\omega_1' \otimes \omega_1'^2 f_1 \otimes \omega_1' f_1 \otimes f_1$	$ax^3\bar{x}^2$
$\omega_1' f_1\omega_1'^2\omega_2' \otimes \omega_1'^2 f_2\omega_1' \otimes f_1\omega_1'^2 \otimes \omega_1' f_1 \otimes f_1$	$ax^2\bar{x}^2$
$\omega_1'^2 f_1\omega_1'\omega_2' \otimes \omega_1'^2 f_2\omega_1' \otimes f_1\omega_1'^2 \otimes \omega_1' f_1 \otimes f_1$	$ax^3\bar{x}^2$
$\omega_1'^3\omega_2' f_1 \otimes \omega_1'^3 f_2 \otimes f_1\omega_1'^2 \otimes \omega_1' f_1 \otimes f_1$	$ax^4\bar{x}^3y$
$\omega_1' f_1\omega_1'^2\omega_2' \otimes \omega_1'^2 f_2\omega_1' \otimes \omega_1' f_1\omega_1' \otimes \omega_1' f_1 \otimes f_1$	$ax^3\bar{x}^2$
$\omega_1' f_1\omega_1'^2\omega_2' \otimes \omega_1'^2 f_2\omega_1' \otimes \omega_1'^2 f_1 \otimes \omega_1' f_1 \otimes f_1$	$ax^4\bar{x}^2$
$\omega_1'^2 f_1\omega_1'\omega_2' \otimes \omega_1'^2 f_2\omega_1' \otimes \omega_1' f_1\omega_1' \otimes \omega_1' f_1 \otimes f_1$	$ax^4\bar{x}^2$
$\omega_1'^3\omega_2' f_1 \otimes \omega_1'^3 f_2 \otimes \omega_1' f_1\omega_1' \otimes \omega_1' f_1 \otimes f_1$	$ax^5\bar{x}^3y$
$\omega_1'^2 f_1\omega_1'\omega_2' \otimes \omega_1'^2 f_2\omega_1' \otimes \omega_1'^2 f_1 \otimes \omega_1' f_1 \otimes f_1$	$ax^5\bar{x}^2$
$\omega_1'^3\omega_2' f_1 \otimes \omega_1'^3 f_2 \otimes \omega_1'^2 f_1 \otimes f_1\omega_1' \otimes f_1$	$ax^5\bar{x}^3y$
$\omega_1'^3\omega_2' f_1 \otimes \omega_1'^3 f_2 \otimes \omega_1'^2 f_1 \otimes \omega_1' f_1 \otimes f_1$	$ax^6\bar{x}^3y$

SUMMANDS	**5**
$\omega_1'^3 f_2\omega_1' \otimes f_1\omega_1'^3 \otimes f_1\omega_1'^2 \otimes f_1\omega_1' \otimes f_1$	$a\bar{x}^3$
$\omega_1'^3 f_2\omega_1' \otimes f_1\omega_1'^3 \otimes \omega_1' f_1\omega_1' \otimes f_1\omega_1' \otimes f_1$	$ax\bar{x}^3$
$\omega_1'^3 f_2\omega_1' \otimes f_1\omega_1'^3 \otimes \omega_1'^2 f_1 \otimes f_1\omega_1' \otimes f_1$	$ax^2\bar{x}^3$

SUMMANDS	**5**
$\omega_1'^3 f_2\omega_1' \otimes \omega_1' f_1\omega_1'^2 \otimes f_1\omega_1'^2 \otimes f_1\omega_1' \otimes f_1$	$ax\bar{x}^3$
$\omega_1'^3 f_2\omega_1' \otimes \omega_1'^2 f_1\omega_1' \otimes f_1\omega_1'^2 \otimes f_1\omega_1' \otimes f_1$	$ax^2\bar{x}^3$
$\omega_1'^3 f_2\omega_1' \otimes \omega_1'^3 f_1 \otimes f_1\omega_1'^2 \otimes f_1\omega_1' \otimes f_1$	$ax^3\bar{x}^3$
$\omega_1'^3 f_2\omega_1' \otimes \omega_1' f_1\omega_1'^2 \otimes \omega_1' f_1\omega_1' \otimes f_1\omega_1' \otimes f_1$	$ax^2\bar{x}^3$
$\omega_1'^3 f_2\omega_1' \otimes \omega_1' f_1\omega_1'^2 \otimes \omega_1'^2 f_1 \otimes f_1\omega_1' \otimes f_1$	$ax^3\bar{x}^3$
$\omega_1'^3 f_2\omega_1' \otimes \omega_1'^2 f_1\omega_1' \otimes \omega_1' f_1\omega_1' \otimes f_1\omega_1' \otimes f_1$	$ax^3\bar{x}^3$
$\omega_1'^3 f_2\omega_1' \otimes \omega_1'^3 f_1 \otimes \omega_1' f_1\omega_1' \otimes f_1\omega_1' \otimes f_1$	$ax^4\bar{x}^3$
$\omega_1'^3 f_2\omega_1' \otimes \omega_1'^2 f_1\omega_1' \otimes \omega_1'^2 f_1 \otimes f_1\omega_1' \otimes f_1$	$ax^4\bar{x}^3$
$\omega_1'^3 f_2\omega_1' \otimes \omega_1'^3 f_1 \otimes \omega_1'^2 f_1 \otimes f_1\omega_1' \otimes f_1$	$ax^5\bar{x}^3$
$\omega_1'^3 f_2\omega_1' \otimes f_1\omega_1'^3 \otimes f_1\omega_1'^2 \otimes \omega_1' f_1 \otimes f_1$	$ax\bar{x}^3$
$\omega_1'^3 f_2\omega_1' \otimes f_1\omega_1'^3 \otimes \omega_1' f_1\omega_1' \otimes \omega_1' f_1 \otimes f_1$	$ax^2\bar{x}^3$
$\omega_1'^3 f_2\omega_1' \otimes f_1\omega_1'^3 \otimes \omega_1'^2 f_1 \otimes \omega_1' f_1 \otimes f_1$	$ax^3\bar{x}^3$
$\omega_1'^3 f_2\omega_1' \otimes \omega_1' f_1\omega_1'^2 \otimes f_1\omega_1'^2 \otimes \omega_1' f_1 \otimes f_1$	$ax^2\bar{x}^3$
$\omega_1'^3 f_2\omega_1' \otimes \omega_1'^2 f_1\omega_1' \otimes f_1\omega_1'^2 \otimes \omega_1' f_1 \otimes f_1$	$ax^3\bar{x}^3$
$\omega_1'^3 f_2\omega_1' \otimes \omega_1'^3 f_1 \otimes f_1\omega_1'^2 \otimes \omega_1' f_1 \otimes f_1$	$ax^4\bar{x}^3$
$\omega_1'^3 f_2\omega_1' \otimes \omega_1' f_1\omega_1'^2 \otimes \omega_1' f_1\omega_1' \otimes \omega_1' f_1 \otimes f_1$	$ax^3\bar{x}^3$
$\omega_1'^3 f_2\omega_1' \otimes \omega_1' f_1\omega_1'^2 \otimes \omega_1'^2 f_1 \otimes \omega_1' f_1 \otimes f_1$	$ax^4\bar{x}^3$
$\omega_1'^3 f_2\omega_1' \otimes \omega_1'^2 f_1\omega_1' \otimes \omega_1' f_1\omega_1' \otimes \omega_1' f_1 \otimes f_1$	$ax^4\bar{x}^3$
$\omega_1'^3 f_2\omega_1' \otimes \omega_1'^3 f_1 \otimes \omega_1' f_1\omega_1' \otimes \omega_1' f_1 \otimes f_1$	$ax^5\bar{x}^3$
$\omega_1'^3 f_2\omega_1' \otimes \omega_1'^2 f_1\omega_1' \otimes \omega_1'^2 f_1 \otimes \omega_1' f_1 \otimes f_1$	$ax^5\bar{x}^3$
$\omega_1'^3 f_2\omega_1' \otimes \omega_1'^3 f_1 \otimes \omega_1'^2 f_1 \otimes \omega_1' f_1 \otimes f_1$	$ax^6\bar{x}^3$

The relevant terms and the results of calculations of $\Delta^{(4)}(X)$ for $X = f_1 f_2 f_1^3$ in $(*)$ are as follows:

TABULAR 5

SUMMANDS	**1**
$f_1\omega_2'\omega_1'^3 \otimes \omega_2' f_1\omega_1'^2 \otimes \omega_2' f_1\omega_1' \otimes \omega_2' f_1 \otimes f_2$	ay^3
$f_1\omega_2'\omega_1'^3 \otimes \omega_2' f_1\omega_1'^2 \otimes \omega_1'\omega_2' f_1 \otimes \omega_2' f_1 \otimes f_2$	axy^3
$f_1\omega_2'\omega_1'^3 \otimes \omega_1'\omega_2' f_1\omega_1' \otimes \omega_2' f_1\omega_1' \otimes \omega_2' f_1 \otimes f_2$	axy^3
$f_1\omega_2'\omega_1'^3 \otimes \omega_1'^2\omega_2' f_1 \otimes \omega_2' f_1\omega_1' \otimes \omega_2' f_1 \otimes f_2$	ax^2y^3
$f_1\omega_2'\omega_1'^3 \otimes \omega_1'\omega_2' f_1\omega_1' \otimes \omega_1'\omega_2' f_1 \otimes \omega_2' f_1 \otimes f_2$	ax^2y^3
$f_1\omega_2'\omega_1'^3 \otimes \omega_1'^2\omega_2' f_1 \otimes \omega_1'\omega_2' f_1 \otimes \omega_2' f_1 \otimes f_2$	ax^3y^3
$\omega_1'\omega_2' f_1\omega_1'^2 \otimes f_1\omega_2'\omega_1'^2 \otimes \omega_2' f_1\omega_1' \otimes \omega_2' f_1 \otimes f_2$	axy^3
$\omega_1'\omega_2' f_1\omega_1'^2 \otimes f_1\omega_2'\omega_1'^2 \otimes \omega_1'\omega_2' f_1 \otimes \omega_2' f_1 \otimes f_2$	ax^2y^3
$\omega_1'^2\omega_2' f_1\omega_1' \otimes f_1\omega_2'\omega_1'^2 \otimes \omega_2' f_1\omega_1' \otimes \omega_2' f_1 \otimes f_2$	ax^2y^3
$\omega_1'^3\omega_2' f_1 \otimes f_1\omega_2'\omega_1'^2 \otimes \omega_2' f_1\omega_1' \otimes \omega_2' f_1 \otimes f_2$	ax^3y^3
$\omega_1'^2\omega_2' f_1\omega_1' \otimes f_1\omega_2'\omega_1'^2 \otimes \omega_1'\omega_2' f_1 \otimes \omega_2' f_1 \otimes f_2$	ax^3y^3
$\omega_1'^3\omega_2' f_1 \otimes f_1\omega_2'\omega_1'^2 \otimes \omega_1'\omega_2' f_1 \otimes \omega_2' f_1 \otimes f_2$	ax^4y^3
$\omega_1'\omega_2' f_1\omega_1'^2 \otimes \omega_1'\omega_2' f_1\omega_1' \otimes f_1\omega_1'\omega_2' \otimes \omega_2' f_1 \otimes f_2$	ax^2y^3
$\omega_1'\omega_2' f_1\omega_1'^2 \otimes \omega_1'^2\omega_2' f_1 \otimes f_1\omega_1'\omega_2' \otimes \omega_2' f_1 \otimes f_2$	ax^3y^3
$\omega_1'^2\omega_2' f_1\omega_1' \otimes \omega_1'\omega_2' f_1\omega_1' \otimes f_1\omega_1'\omega_2' \otimes \omega_2' f_1 \otimes f_2$	ax^3y^3

SUMMANDS	**1**
$\omega_1^{\prime 3}\omega'_2 f_1 \otimes \omega'_1\omega'_2 f_1\omega'_1 \otimes f_1\omega'_1\omega'_2 \otimes \omega'_2 f_1 \otimes f_2$	ax^4y^3
$\omega_1^{\prime 2}\omega'_2 f_1\omega'_1 \otimes \omega_1^{\prime 2}\omega'_2 f_1 \otimes f_1\omega'_1\omega'_2 \otimes \omega'_2 f_1 \otimes f_2$	ax^4y^3
$\omega_1^{\prime 3}\omega'_2 f_1 \otimes \omega_1^{\prime 2}\omega'_2 f_1 \otimes f_1\omega'_1\omega'_2 \otimes \omega'_2 f_1 \otimes f_2$	ax^5y^3
$\omega'_1\omega'_2 f_1\omega_1^{\prime 2} \otimes \omega'_1\omega'_2 f_1\omega'_1 \otimes \omega'_1\omega'_2 f_1 \otimes f_1\omega'_2 \otimes f_2$	ax^3y^3
$\omega'_1\omega'_2 f_1\omega_1^{\prime 2} \otimes \omega_1^{\prime 2}\omega'_2 f_1 \otimes \omega'_1\omega'_2 f_1 \otimes f_1\omega'_2 \otimes f_2$	ax^4y^3
$\omega_1^{\prime 2}\omega'_2 f_1\omega'_1 \otimes \omega'_1\omega'_2 f_1\omega'_1 \otimes \omega'_1\omega'_2 f_1 \otimes f_1\omega'_2 \otimes f_2$	ax^4y^3
$\omega_1^{\prime 3}\omega'_2 f_1 \otimes \omega'_1\omega'_2 f_1\omega'_1 \otimes \omega'_1\omega'_2 f_1 \otimes f_1\omega'_2 \otimes f_2$	ax^5y^3
$\omega_1^{\prime 2}\omega'_2 f_1\omega'_1 \otimes \omega_1^{\prime 2}\omega'_2 f_1 \otimes \omega'_1\omega'_2 f_1 \otimes f_1\omega'_2 \otimes f_2$	ax^5y^3
$\omega_1^{\prime 3}\omega'_2 f_1 \otimes \omega_1^{\prime 2}\omega'_2 f_1 \otimes \omega'_1\omega'_2 f_1 \otimes f_1\omega'_2 \otimes f_2$	ax^6y^3

SUMMANDS	**2**
$f_1\omega'_2\omega_1^{\prime 3} \otimes \omega'_2 f_1\omega_1^{\prime 2} \otimes \omega'_2 f_1\omega'_1 \otimes f_2\omega'_1 \otimes f_1$	ay^2
$f_1\omega'_2\omega_1^{\prime 3} \otimes \omega'_2 f_1\omega_1^{\prime 2} \otimes \omega'_1\omega'_2 f_1 \otimes f_2\omega'_1 \otimes f_1$	axy^2
$f_1\omega'_2\omega_1^{\prime 3} \otimes \omega'_1\omega'_2 f_1\omega'_1 \otimes \omega'_2 f_1\omega'_1 \otimes f_2\omega'_1 \otimes f_1$	axy^2
$f_1\omega'_2\omega_1^{\prime 3} \otimes \omega_1^{\prime 2}\omega'_2 f_1 \otimes \omega'_2 f_1\omega'_1 \otimes f_2\omega'_1 \otimes f_1$	ax^2y^2
$f_1\omega'_2\omega_1^{\prime 3} \otimes \omega'_1\omega'_2 f_1\omega'_1 \otimes \omega'_1\omega'_2 f_1 \otimes f_2\omega'_1 \otimes f_1$	ax^2y^2
$f_1\omega'_2\omega_1^{\prime 3} \otimes \omega_1^{\prime 2}\omega'_2 f_1 \otimes \omega'_1\omega'_2 f_1 \otimes f_2\omega'_1 \otimes f_1$	ax^3y^2
$\omega'_1\omega'_2 f_1\omega_1^{\prime 2} \otimes f_1\omega'_2\omega_1^{\prime 2} \otimes \omega'_2 f_1\omega'_1 \otimes f_2\omega'_1 \otimes f_1$	axy^2
$\omega'_1\omega'_2 f_1\omega_1^{\prime 2} \otimes f_1\omega'_2\omega_1^{\prime 2} \otimes \omega'_1\omega'_2 f_1 \otimes f_2\omega'_1 \otimes f_1$	ax^2y^2
$\omega_1^{\prime 2}\omega'_2 f_1\omega'_1 \otimes f_1\omega'_2\omega_1^{\prime 2} \otimes \omega'_2 f_1\omega'_1 \otimes f_2\omega'_1 \otimes f_1$	ax^2y^2
$\omega_1^{\prime 3}\omega'_2 f_1 \otimes f_1\omega'_2\omega_1^{\prime 2} \otimes \omega'_2 f_1\omega'_1 \otimes f_2\omega'_1 \otimes f_1$	ax^3y^2
$\omega_1^{\prime 2}\omega'_2 f_1\omega'_1 \otimes f_1\omega'_2\omega_1^{\prime 2} \otimes \omega'_1\omega'_2 f_1 \otimes f_2\omega'_1 \otimes f_1$	ax^3y^2
$\omega_1^{\prime 3}\omega'_2 f_1 \otimes f_1\omega'_2\omega_1^{\prime 2} \otimes \omega'_1\omega'_2 f_1 \otimes f_2\omega'_1 \otimes f_1$	ax^4y^2
$\omega'_1\omega'_2 f_1\omega_1^{\prime 2} \otimes \omega'_1\omega'_2 f_1\omega'_1 \otimes f_1\omega'_1\omega'_2 \otimes f_2\omega'_1 \otimes f_1$	ax^2y^2
$\omega'_1\omega'_2 f_1\omega_1^{\prime 2} \otimes \omega_1^{\prime 2}\omega'_2 f_1 \otimes f_1\omega'_1\omega'_2 \otimes f_2\omega'_1 \otimes f_1$	ax^3y^2
$\omega_1^{\prime 2}\omega'_2 f_1\omega'_1 \otimes \omega'_1\omega'_2 f_1\omega'_1 \otimes f_1\omega'_1\omega'_2 \otimes f_2\omega'_1 \otimes f_1$	ax^3y^2
$\omega_1^{\prime 3}\omega'_2 f_1 \otimes \omega'_1\omega'_2 f_1\omega'_1 \otimes f_1\omega'_1\omega'_2 \otimes f_2\omega'_1 \otimes f_1$	ax^4y^2
$\omega_1^{\prime 2}\omega'_2 f_1\omega'_1 \otimes \omega_1^{\prime 2}\omega'_2 f_1 \otimes f_1\omega'_1\omega'_2 \otimes f_2\omega'_1 \otimes f_1$	ax^4y^2
$\omega_1^{\prime 3}\omega'_2 f_1 \otimes \omega_1^{\prime 2}\omega'_2 f_1 \otimes f_1\omega'_1\omega'_2 \otimes f_2\omega'_1 \otimes f_1$	ax^5y^2
$\omega'_1\omega'_2 f_1\omega_1^{\prime 2} \otimes \omega'_1\omega'_2 f_1\omega'_1 \otimes \omega'_1\omega'_2 f_1 \otimes \omega'_1 f_2 \otimes f_1$	$ax^3\bar{x}y^3$
$\omega'_1\omega'_2 f_1\omega_1^{\prime 2} \otimes \omega_1^{\prime 2}\omega'_2 f_1 \otimes \omega'_1\omega'_2 f_1 \otimes \omega'_1 f_2 \otimes f_1$	$ax^4\bar{x}y^3$
$\omega_1^{\prime 2}\omega'_2 f_1\omega'_1 \otimes \omega'_1\omega'_2 f_1\omega'_1 \otimes \omega'_1\omega'_2 f_1 \otimes \omega'_1 f_2 \otimes f_1$	$ax^4\bar{x}y^3$
$\omega_1^{\prime 3}\omega'_2 f_1 \otimes \omega'_1\omega'_2 f_1\omega'_1 \otimes \omega'_1\omega'_2 f_1 \otimes \omega'_1 f_2 \otimes f_1$	$ax^5\bar{x}y^3$
$\omega_1^{\prime 2}\omega'_2 f_1\omega'_1 \otimes \omega_1^{\prime 2}\omega'_2 f_1 \otimes \omega'_1\omega'_2 f_1 \otimes \omega'_1 f_2 \otimes f_1$	$ax^5\bar{x}y^3$
$\omega_1^{\prime 3}\omega'_2 f_1 \otimes \omega_1^{\prime 2}\omega'_2 f_1 \otimes \omega'_1\omega'_2 f_1 \otimes \omega'_1 f_2 \otimes f_1$	$ax^6\bar{x}y^3$

SUMMANDS	**3**
$f_1\omega'_2\omega_1^{\prime 3} \otimes \omega'_2 f_1\omega_1^{\prime 2} \otimes f_2\omega_1^{\prime 2} \otimes f_1\omega'_1 \otimes f_1$	ay
$f_1\omega'_2\omega_1^{\prime 3} \otimes \omega'_2 f_1\omega_1^{\prime 2} \otimes f_2\omega_1^{\prime 2} \otimes \omega'_1 f_1 \otimes f_1$	axy
$f_1\omega'_2\omega_1^{\prime 3} \otimes \omega'_1\omega'_2 f_1\omega'_1 \otimes f_2\omega_1^{\prime 2} \otimes f_1\omega'_1 \otimes f_1$	axy
$f_1\omega'_2\omega_1^{\prime 3} \otimes \omega_1^{\prime 2}\omega'_2 f_1 \otimes f_2\omega_1^{\prime 2} \otimes f_1\omega'_1 \otimes f_1$	ax^2y
$f_1\omega'_2\omega_1^{\prime 3} \otimes \omega'_1\omega'_2 f_1\omega'_1 \otimes f_2\omega_1^{\prime 2} \otimes \omega'_1 f_1 \otimes f_1$	ax^2y
$f_1\omega'_2\omega_1^{\prime 3} \otimes \omega_1^{\prime 2}\omega'_2 f_1 \otimes f_2\omega_1^{\prime 2} \otimes \omega'_1 f_1 \otimes f_1$	ax^3y

SUMMANDS	**3**
$\omega_1'\omega_2' f_1\omega_1'^2 \otimes f_1\omega_2'\omega_1'^2 \otimes f_2\omega_1'^2 \otimes f_1\omega_1' \otimes f_1$	axy
$\omega_1'\omega_2' f_1\omega_1'^2 \otimes f_1\omega_2'\omega_1'^2 \otimes f_2\omega_1'^2 \otimes \omega_1' f_1 \otimes f_1$	ax^2y
$\omega_1'^2\omega_2' f_1\omega_1' \otimes f_1\omega_2'\omega_1'^2 \otimes f_2\omega_1'^2 \otimes f_1\omega_1' \otimes f_1$	ax^2y
$\omega_1'^3\omega_2' f_1 \otimes f_1\omega_2'\omega_1'^2 \otimes f_2\omega_1'^2 \otimes f_1\omega_1' \otimes f_1$	ax^3y
$\omega_1'^2\omega_2' f_1\omega_1' \otimes f_1\omega_2'\omega_1'^2 \otimes f_2\omega_1'^2 \otimes \omega_1' f_1 \otimes f_1$	ax^3y
$\omega_1'^3\omega_2' f_1 \otimes f_1\omega_2'\omega_1'^2 \otimes f_2\omega_1'^2 \otimes \omega_1' f_1 \otimes f_1$	ax^4y
$\omega_1'\omega_2' f_1\omega_1'^2 \otimes \omega_1'\omega_2' f_1\omega_1' \otimes \omega_1' f_2\omega_1' \otimes f_1\omega_1' \otimes f_1$	$ax^2\bar{x}y^2$
$\omega_1'\omega_2' f_1\omega_1'^2 \otimes \omega_1'^2\omega_2' f_1 \otimes \omega_1' f_2\omega_1' \otimes f_1\omega_1' \otimes f_1$	$ax^3\bar{x}y^2$
$\omega_1'^2\omega_2' f_1\omega_1' \otimes \omega_1'\omega_2' f_1\omega_1' \otimes \omega_1' f_2\omega_1' \otimes f_1\omega_1' \otimes f_1$	$ax^3\bar{x}y^2$
$\omega_1'^3\omega_2' f_1 \otimes \omega_1'\omega_2' f_1\omega_1' \otimes \omega_1' f_2\omega_1' \otimes f_1\omega_1' \otimes f_1$	$ax^4\bar{x}y^2$
$\omega_1'^2\omega_2' f_1\omega_1' \otimes \omega_1'^2\omega_2' f_1 \otimes \omega_1' f_2\omega_1' \otimes f_1\omega_1' \otimes f_1$	$ax^4\bar{x}y^2$
$\omega_1'^3\omega_2' f_1 \otimes \omega_1'^2\omega_2' f_1 \otimes \omega_1' f_2\omega_1' \otimes f_1\omega_1' \otimes f_1$	$ax^5\bar{x}y^2$
$\omega_1'\omega_2' f_1\omega_1'^2 \otimes \omega_1'\omega_2' f_1\omega_1' \otimes \omega_1' f_2\omega_1' \otimes \omega_1' f_1 \otimes f_1$	$ax^3\bar{x}y^2$
$\omega_1'\omega_2' f_1\omega_1'^2 \otimes \omega_1'^2\omega_2' f_1 \otimes \omega_1' f_2\omega_1' \otimes \omega_1' f_1 \otimes f_1$	$ax^4\bar{x}y^2$
$\omega_1'^2\omega_2' f_1\omega_1' \otimes \omega_1'\omega_2' f_1\omega_1' \otimes \omega_1' f_2\omega_1' \otimes \omega_1' f_1 \otimes f_1$	$ax^4\bar{x}y^2$
$\omega_1'^3\omega_2' f_1 \otimes \omega_1'\omega_2' f_1\omega_1' \otimes \omega_1' f_2\omega_1' \otimes \omega_1' f_1 \otimes f_1$	$ax^5\bar{x}y^2$
$\omega_1'^2\omega_2' f_1\omega_1' \otimes \omega_1'^2\omega_2' f_1 \otimes \omega_1' f_2\omega_1' \otimes \omega_1' f_1 \otimes f_1$	$ax^5\bar{x}y^2$
$\omega_1'^3\omega_2' f_1 \otimes \omega_1'^2\omega_2' f_1 \otimes \omega_1' f_2\omega_1' \otimes \omega_1' f_1 \otimes f_1$	$ax^6\bar{x}y^2$

SUMMANDS	**4**
$f_1\omega_2'\omega_1'^3 \otimes f_2\omega_1'^3 \otimes f_1\omega_1'^2 \otimes f_1\omega_1' \otimes f_1$	a
$f_1\omega_2'\omega_1'^3 \otimes f_2\omega_1'^3 \otimes f_1\omega_1'^2 \otimes \omega_1' f_1 \otimes f_1$	ax
$f_1\omega_2'\omega_1'^3 \otimes f_2\omega_1'^3 \otimes \omega_1' f_1\omega_1' \otimes f_1\omega_1' \otimes f_1$	ax
$f_1\omega_2'\omega_1'^3 \otimes f_2\omega_1'^3 \otimes \omega_1'^2 f_1 \otimes f_1\omega_1' \otimes f_1$	ax^2
$f_1\omega_2'\omega_1'^3 \otimes f_2\omega_1'^3 \otimes \omega_1' f_1\omega_1' \otimes \omega_1' f_1 \otimes f_1$	ax^2
$f_1\omega_2'\omega_1'^3 \otimes f_2\omega_1'^3 \otimes \omega_1'^2 f_1 \otimes \omega_1' f_1 \otimes f_1$	ax^3
$\omega_1'\omega_2' f_1\omega_1'^2 \otimes \omega_1' f_2\omega_1'^2 \otimes f_1\omega_1'^2 \otimes f_1\omega_1' \otimes f_1$	$ax\bar{x}y$
$\omega_1'\omega_2' f_1\omega_1'^2 \otimes \omega_1' f_2\omega_1'^2 \otimes f_1\omega_1'^2 \otimes \omega_1' f_1 \otimes f_1$	$ax^2\bar{x}y$
$\omega_1'^2\omega_2' f_1\omega_1' \otimes \omega_1' f_2\omega_1'^2 \otimes f_1\omega_1'^2 \otimes f_1\omega_1' \otimes f_1$	$ax^2\bar{x}y$
$\omega_1'^3\omega_2' f_1 \otimes \omega_1' f_2\omega_1'^2 \otimes f_1\omega_1'^2 \otimes f_1\omega_1' \otimes f_1$	$ax^3\bar{x}y$
$\omega_1'^2\omega_2' f_1\omega_1' \otimes \omega_1' f_2\omega_1'^2 \otimes f_1\omega_1'^2 \otimes \omega_1' f_1 \otimes f_1$	$ax^3\bar{x}y$
$\omega_1'^3\omega_2' f_1 \otimes \omega_1' f_2\omega_1'^2 \otimes f_1\omega_1'^2 \otimes \omega_1' f_1 \otimes f_1$	$ax^4\bar{x}y$
$\omega_1'\omega_2' f_1\omega_1'^2 \otimes \omega_1' f_2\omega_1'^2 \otimes \omega_1' f_1\omega_1' \otimes f_1\omega_1' \otimes f_1$	$ax^2\bar{x}y$
$\omega_1'\omega_2' f_1\omega_1'^2 \otimes \omega_1' f_2\omega_1'^2 \otimes \omega_1'^2 f_1 \otimes f_1\omega_1' \otimes f_1$	$ax^3\bar{x}y$
$\omega_1'^2\omega_2' f_1\omega_1' \otimes \omega_1' f_2\omega_1'^2 \otimes \omega_1' f_1\omega_1' \otimes f_1\omega_1' \otimes f_1$	$ax^3\bar{x}y$
$\omega_1'^3\omega_2' f_1 \otimes \omega_1' f_2\omega_1'^2 \otimes \omega_1' f_1\omega_1' \otimes f_1\omega_1' \otimes f_1$	$ax^4\bar{x}y$
$\omega_1'^2\omega_2' f_1\omega_1' \otimes \omega_1' f_2\omega_1'^2 \otimes \omega_1'^2 f_1 \otimes f_1\omega_1' \otimes f_1$	$ax^4\bar{x}y$
$\omega_1'^3\omega_2' f_1 \otimes \omega_1' f_2\omega_1'^2 \otimes \omega_1'^2 f_1 \otimes f_1\omega_1' \otimes f_1$	$ax^5\bar{x}y$
$\omega_1'\omega_2' f_1\omega_1'^2 \otimes \omega_1' f_2\omega_1'^2 \otimes \omega_1' f_1\omega_1' \otimes \omega_1' f_1 \otimes f_1$	$ax^3\bar{x}y$
$\omega_1'\omega_2' f_1\omega_1'^2 \otimes \omega_1' f_2\omega_1'^2 \otimes \omega_1'^2 f_1 \otimes \omega_1' f_1 \otimes f_1$	$ax^4\bar{x}y$
$\omega_1'^2\omega_2' f_1\omega_1' \otimes \omega_1' f_2\omega_1'^2 \otimes \omega_1' f_1\omega_1' \otimes \omega_1' f_1 \otimes f_1$	$ax^4\bar{x}y$
$\omega_1'^3\omega_2' f_1 \otimes \omega_1' f_2\omega_1'^2 \otimes \omega_1' f_1\omega_1' \otimes \omega_1' f_1 \otimes f_1$	$ax^5\bar{x}y$

SUMMANDS	**4**
$\omega_1'^2\omega_2' f_1\omega_1' \otimes \omega_1' f_2\omega_1'^2 \otimes \omega_1'^2 f_1 \otimes \omega_1' f_1 \otimes f_1$	$ax^5\bar{x}y$
$\omega_1'^3\omega_2' f_1 \otimes \omega_1' f_2\omega_1'^2 \otimes \omega_1'^2 f_1 \otimes \omega_1' f_1 \otimes f_1$	$ax^6\bar{x}y$

SUMMANDS	**5**
$\omega_1' f_2\omega_1'^3 \otimes f_1\omega_1'^3 \otimes f_1\omega_1'^2 \otimes f_1\omega_1' \otimes f_1$	$a\bar{x}$
$\omega_1' f_2\omega_1'^3 \otimes f_1\omega_1'^3 \otimes f_1\omega_1'^2 \otimes \omega_1' f_1 \otimes f_1$	$ax\bar{x}$
$\omega_1' f_2\omega_1'^3 \otimes f_1\omega_1'^3 \otimes \omega_1' f_1\omega_1' \otimes f_1\omega_1' \otimes f_1$	$ax\bar{x}$
$\omega_1' f_2\omega_1'^3 \otimes f_1\omega_1'^3 \otimes \omega_1'^2 f_1 \otimes f_1\omega_1' \otimes f_1$	$ax^2\bar{x}$
$\omega_1' f_2\omega_1'^3 \otimes f_1\omega_1'^3 \otimes \omega_1' f_1\omega_1' \otimes \omega_1' f_1 \otimes f_1$	$ax^2\bar{x}$
$\omega_1' f_2\omega_1'^3 \otimes f_1\omega_1'^3 \otimes \omega_1'^2 f_1 \otimes \omega_1' f_1 \otimes f_1$	$ax^3\bar{x}$
$\omega_1' f_2\omega_1'^3 \otimes \omega_1' f_1\omega_1'^2 \otimes f_1\omega_1'^2 \otimes f_1\omega_1' \otimes f_1$	$ax\bar{x}$
$\omega_1' f_2\omega_1'^3 \otimes \omega_1' f_1\omega_1'^2 \otimes f_1\omega_1'^2 \otimes \omega_1' f_1 \otimes f_1$	$ax^2\bar{x}$
$\omega_1' f_2\omega_1'^3 \otimes \omega_1'^2 f_1\omega_1' \otimes f_1\omega_1'^2 \otimes f_1\omega_1' \otimes f_1$	$ax^2\bar{x}$
$\omega_1' f_2\omega_1'^3 \otimes \omega_1'^3 f_1 \otimes f_1\omega_1'^2 \otimes f_1\omega_1' \otimes f_1$	$ax^3\bar{x}$
$\omega_1' f_2\omega_1'^3 \otimes \omega_1'^2 f_1\omega_1' \otimes f_1\omega_1'^2 \otimes \omega_1' f_1 \otimes f_1$	$ax^3\bar{x}$
$\omega_1' f_2\omega_1'^3 \otimes \omega_1'^3 f_1 \otimes f_1\omega_1'^2 \otimes \omega_1' f_1 \otimes f_1$	$ax^4\bar{x}$
$\omega_1' f_2\omega_1'^3 \otimes \omega_1' f_1\omega_1'^2 \otimes \omega_1' f_1\omega_1' \otimes f_1\omega_1' \otimes f_1$	$ax^2\bar{x}$
$\omega_1' f_2\omega_1'^3 \otimes \omega_1' f_1\omega_1'^2 \otimes \omega_1'^2 f_1 \otimes f_1\omega_1' \otimes f_1$	$ax^3\bar{x}$
$\omega_1' f_2\omega_1'^3 \otimes \omega_1'^2 f_1\omega_1' \otimes \omega_1' f_1\omega_1' \otimes f_1\omega_1' \otimes f_1$	$ax^3\bar{x}$
$\omega_1' f_2\omega_1'^3 \otimes \omega_1'^3 f_1 \otimes \omega_1' f_1\omega_1' \otimes f_1\omega_1' \otimes f_1$	$ax^4\bar{x}$
$\omega_1' f_2\omega_1'^3 \otimes \omega_1'^2 f_1\omega_1' \otimes \omega_1'^2 f_1 \otimes f_1\omega_1' \otimes f_1$	$ax^4\bar{x}$
$\omega_1' f_2\omega_1'^3 \otimes \omega_1'^3 f_1 \otimes \omega_1'^2 f_1 \otimes f_1\omega_1' \otimes f_1$	$ax^5\bar{x}$
$\omega_1' f_2\omega_1'^3 \otimes \omega_1' f_1\omega_1'^2 \otimes \omega_1' f_1\omega_1' \otimes \omega_1' f_1 \otimes f_1$	$ax^3\bar{x}$
$\omega_1' f_2\omega_1'^3 \otimes \omega_1' f_1\omega_1'^2 \otimes \omega_1'^2 f_1 \otimes \omega_1' f_1 \otimes f_1$	$ax^4\bar{x}$
$\omega_1' f_2\omega_1'^3 \otimes \omega_1'^2 f_1\omega_1' \otimes \omega_1' f_1\omega_1' \otimes \omega_1' f_1 \otimes f_1$	$ax^4\bar{x}$
$\omega_1' f_2\omega_1'^3 \otimes \omega_1'^3 f_1 \otimes \omega_1' f_1\omega_1' \otimes \omega_1' f_1 \otimes f_1$	$ax^5\bar{x}$
$\omega_1' f_2\omega_1'^3 \otimes \omega_1'^2 f_1\omega_1' \otimes \omega_1'^2 f_1 \otimes \omega_1' f_1 \otimes f_1$	$ax^5\bar{x}$
$\omega_1' f_2\omega_1'^3 \otimes \omega_1'^3 f_1 \otimes \omega_1'^2 f_1 \otimes \omega_1' f_1 \otimes f_1$	$ax^6\bar{x}$

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